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Nonlinear oscillations, bifurcations and chaos of functionally graded materials plate

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Abstract

An analysis on the nonlinear dynamics of a simply supported functionally graded materials (FGMs) rectangular plate subjected to the transversal and in-plane excitations is presented in a thermal environment for the first time. Material properties are assumed to be temperature dependent. Based on Reddy's third-order plate theory, the nonlinear governing equations of motion for the FGM plates are derived using Hamilton's principle. Galerkin's method is utilized to discretize the governing partial equations to a two-degree-of-freedom nonlinear system including the quadratic and cubic nonlinear terms under combined parametric and external excitations. The resonant case considered here is 1:1 internal resonance and principal parametric resonance. The asymptotic perturbation method is utilized to obtain four-dimensional nonlinear averaged equation. The numerical method is used to find the nonlinear dynamic responses of the FGM rectangular plate. It was found that periodic, quasi-periodic solutions and chaotic motions exist for the FGM rectangular plates under certain conditions. It is believed that the forcing excitations f_1 and f_2 can change the form of motions for the FGM rectangular plate.

1. Introduction

Functionally graded materials (FGMs) are new engineering composite materials, which are being widely applied to large space station, shuttle, aircraft, automotive and many others in recent years [1,2]. The FGMs are microscopically inhomogeneous composites usually made from a mixture of metals and ceramics. By gradually varying the volume fraction of constituent materials, their material properties exhibit a smooth and continuous change from one surface to another. Thus, interface problems and mitigating thermal stress concentrations can be eliminated. In the FGMs, the micro-structures are spatially varied through non-uniform distribution of the reinforcement phases using the reinforcement with different properties, sizes and shapes as well as by interchanging the roles of the reinforcement and matrix phases in a continuous manner [3]. With the increasing use of FGM plates in engineering fields, research on the nonlinear dynamics, bifurcations, and

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chaos of the FGM plates plays a significant role in applications. However, to date, only few studies on the bifurcations and chaos of the FGM plates have been conducted.

In the last 10 years, several researchers have focussed their attention on investigating the dynamics the FGM plates. Praveen and Reddy [4] provided the nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates subjected to pressure loading and thickness varying temperature fields. Cheng and Batra [5] used Reddy's third-order plate theory to study buckling and steady-state vibrations of a simply supported functionally gradient isotropic polygonal plate resting on a Winkler Pasternak elastic foundation and subjected to uniform in-plane hydrostatic loads. Ng et al. [6] analyzed the parametric resonance of functionally graded rectangular plates under harmonic in-plane loading. Yang and Shen [7] studied the dynamic responses of initially stressed FGM rectangular thin plates subjected to partially distributed impulsive lateral loads and without or resting on an elastic foundation. In 2002, they [8] investigated the free and forced vibrations of FGM plates in a thermal environment. The material properties were assumed to be temperature-dependent and graded in the thickness direction according to a simple power-law distribution in terms of the volume fractions of the constituents. Based on Reddy's higher-order shear deformation shell theory, Yang and Shen [9] analyzed the free vibration and parametric resonance of shear deformable functionally graded cylindrical panels subjected to combined static and periodic axial forces in a thermal environment. Yang et al. [10] investigated the large amplitude vibration of pre-stressed FGM laminated plates that are composed of a shear deformable functionally graded layer and two surface-mounted piezoelectric actuator layers.

In addition, Qian et al. [11] analyzed the static deformations, free and forced vibrations of a thick functionally graded elastic rectangular plate by using a higher-order shear and normal deformable plate theory and the Petrov-Galerkin method. Senthil and Batra [12] gave a three-dimensional exact solution for free and forced vibrations of simply supported FGMs rectangular plates. Huang and Shen [13] studied the nonlinear vibrations and dynamic responses of FGM plates in thermal environment, in which the heat conduction and temperature-dependent material properties were both considered. Chen [14] investigated the nonlinear vibrations of FGM plates with arbitrary initial stresses. The effect of transverse shear strains and rotary inertia was included in five partial differential governing equations.

Studies of the nonlinear vibrations and dynamic stability of the plate and shells have been extensively conducted in the past two decades. Many of these studies are focused on isotropic or laminated composite plates and shells. Hadian and Nayfeh [15] used the method of multiple scales to analyze asymmetric nonlinear responses of clamped circular plates subjected to harmonic excitations and considered the case of a combination-type internal resonance. Chang et al. [16] investigated the bifurcations and chaos of a rectangular thin plate with 1:1 internal resonance. Anlas and Elbeyli [17] studied the nonlinear dynamics of a simply supported rectangular plate subjected to transverse harmonic excitation. Zhang et al. [18] investigated the global bifurcations and chaotic dynamics of a parametrically and externally excited simply supported rectangular thin plate. Recently, Ye et al. [19] studied the local and global nonlinear dynamics of a parametrically excited rectangular symmetric cross-ply laminated composite plate. In addition, Zhang et al. [20] gave further studies on the nonlinear oscillations and chaos of a rectangular symmetric cross-by laminated plate under parametric excitation.

In the last century, to investigate the nonlinear oscillations, many asymptotic perturbation techniques, such as the averaging method, the Krylov, Bogoliubov and Mitropolsky (KBM) method, the method of multiple scales and the harmonic balance method, have been presented and widely used to construct the approximate solutions of weakly nonlinear systems. In general situations, analysis is carried out only up to the first-order approximation since higher-order terms do not have large influence on the qualitative characteristics of the asymptotic solutions. However, the quadratic nonlinearities cannot be included in the first-order approximate solutions when they exist in the original nonlinear systems. Therefore, in order to acquire better qualitative and quantitative characteristics of nonlinear systems having the quadratic and cubic nonlinearities, the second-order averaging or perturbation procedure should be considered. In order to investigate conveniently nonlinear dynamic responses of systems including the quadratic and cubic nonlinearities, an asymptotic perturbation method was developed by Maccari [21–25] based on the slow temporal rescaling and balancing of the harmonic terms with a simple iteration. In the certain sense, the asymptotic perturbation method can be considered as an attempt to link the most useful characteristics of the harmonic balance and the method of multiple scales. Recently, Ye et al. [26] utilized the asymptotic perturbation method to study the nonlinear oscillations and chaotic dynamics of an antisymmetric cross-ply laminated composite rectangular thin plate under parametric excitation.

This paper focuses on research on the bifurcations and chaotic dynamics of a simply supported at the four edges, FGM rectangular plate subjected to the in-plane and transversal excitations simultaneously in the uniform thermal environment. Material properties of the constituents are graded in the thickness direction according to a power-law distribution. In the framework of Reddy's third-order shear deformation plate theory [27], the governing equations of motion for the FGMs rectangular plate are derived using Hamilton's principle. Because only transverse nonlinear oscillations of the FGM plate are considered, the equations of motion can be reduced into a two-degree-of-freedom nonlinear system under combined parametric and external excitations using Galerkin's method. The case of 1:1 internal resonance and primary parametric resonance is considered for the FGM rectangular plate. The asymptotic perturbation method developed by Maccari [21–25] is employed to transfer the second-order nonautonomous nonlinear differential equation with the quadratic and cubic nonlinear terms to the first-order nonlinear averaged equation. Using the numerical method, the averaged equation is analyzed to find the nonlinear responses and chaotic motions of the FGM rectangular plate.

2. Formulation

A simply supported at the four-edges FGMs rectangular plate subjected to the in-plane and transversal excitations is considered, as shown in Fig. 1. The edge width and length of the FGM rectangular plate in the x and y directions are, respectively, a and b and the thickness is h. A Cartesian coordinate $Oxyz$ is located in the middle surface of the FGMs rectangular plate. Assume that (u, v, w) and (u_0, v_0, w_0) represent the displacements of an arbitrary point and a point in the middle surface of the FGMs rectangular plate in the x, y and z directions, respectively. It is also assumed that ϕ_x and ϕ_y, respectively, represent the mid-plane rotations of two transverse normals about the x-and y-axis. The in-plane excitation of the FGMs plate is distributed along the y direction at $x = 0$ and a and is of the form $-(p_0 - p_1 \cos \Omega_2 t)$. The transversal excitation subject to the FGMs plate is represented by $F(x,y)\cos \Omega_1 t$. Here, Ω_1 and Ω_2 are the frequencies of the transversal excitation and the in-plane excitation, respectively.

2.1. Materials properties

Generally speaking, most of the FGMs are employed in high-temperature environment and many of the constituent materials may possess temperature-dependent properties. It is assumed that the plate is made from a mixture of ceramics and metals with continuous varying such that the bottom surface of the plate is metal-rich, whereas the top surface is cerami-rich. The material properties P, such as Young's modulus E and the coefficient α of thermal expansion, can be expressed as a function of the temperature [28,29]

$$P_i = P_0(P_{-1}T^{-1} + 1 + P_1 T + P_2 T^2 + P_3 T^3), \tag{1}$$

where P_0, P_{-1}, P_1, P_2 and P_3 are temperature coefficients.

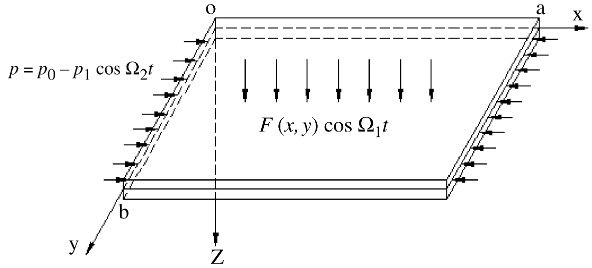

Fig. 1. The model of an FGM rectangular plate and the coordinate system.

The effective material properties P of the FGMs can be expressed as

$$P = P_t V_c + P_b V_m, \tag{2}$$

where subscripts t and b, respectively, represent the top and bottom surfaces of the FGMs plate, V_c and V_m are the ceramic and metal volume fractions and add to unity

$$V_c + V_m = 1. \tag{3}$$

The metal volume fraction V_m is defined as [37]

$$V_m(z) = \left(\frac{2z+h}{2h}\right)^N, \tag{4}$$

where power-law exponent N is a real number that characterizes the metal variation profile through the plate thickness.

From Eqs. (2)–(4), Young's modulus E, the coefficient α of the thermal expansion, the mass density ρ and the thermal conductivity κ can be expressed as

$$E = (E_b - E_t)V_m + E_t, \tag{5a}$$

$$\alpha = (\alpha_b - \alpha_t)V_m + \alpha_t, \tag{5b}$$

$$\rho = (\rho_b - \rho_t)V_m + \rho_t, \tag{5c}$$

$$\kappa = (\kappa_b - \kappa_t)V_m + \kappa_t. \tag{5d}$$

We assume that the temperature variation occurs in the thickness direction only and the one-dimensional temperature field is constant in the xy plane of the plate. In this case, the temperature distribution along the thickness of the plate can be obtained by solving a steady-state heat transfer equation

$$-\frac{\mathrm{d}}{\mathrm{d}z}\left[\kappa(z)\frac{\mathrm{d}T}{\mathrm{d}z}\right] = 0. \tag{6}$$

This equation is solved by imposing the boundary condition of $T = T_t$ at $z = h/2$ and $T = T_b$ at $z = -h/2$. For an isotropic material, the solution of Eq. (6) may be expressed as

$$T(z) = \frac{T_t + T_b}{2} + \frac{T_t - T_b}{h} z. \tag{7}$$

It is also supposed that the FGMs plate is linear elastic throughout the deformation, and that the plate is initially stress-free at T_0 and is subjected to a uniform temperature variation $\Delta T = T - T_0$.

2.2. Equations of motion

According to Reddy's third-order shear deformation (TSDT) in Refs. [27,30–32], the displacement field of the FGMs plate is assumed to be

$$\begin{aligned} u(x,y,t) &= u_0(x,y,t) + z\phi_x(x,y,t) - c_1 z^3\left(\phi_x + \frac{\partial w_0}{\partial x}\right), \\ v(x,y,t) &= v_0(x,y,t) + z\phi_y(x,y,t) - c_1 z^3\left(\phi_y + \frac{\partial w_0}{\partial y}\right), \\ w(x,y,t) &= w_0(x,y,t). \end{aligned} \tag{8}$$

Based on the nonlinear strain–displacement relation and the above displacement field, we obtain

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \quad \varepsilon_{yy} = \frac{\partial v}{\partial y} + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \quad \gamma_{xy} = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right), \\ &\gamma_{yz} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right), \quad \gamma_{zx} = \frac{1}{2}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \end{aligned} \tag{9}$$

and

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} + z \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = \begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{zx}^{(0)} \end{Bmatrix} + z^2 \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{zx}^{(2)} \end{Bmatrix}, \tag{10}$$

where

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u_0}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial x}\right)^2 \\ \dfrac{\partial v_0}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w_0}{\partial y}\right)^2 \\ \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} + \dfrac{\partial w_0}{\partial x}\dfrac{\partial w_0}{\partial y} \end{Bmatrix}, \quad \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = -c_1 \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix},$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = -c_1 \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} + \dfrac{\partial^2 w_0}{\partial x^2} \\ \dfrac{\partial \phi_y}{\partial y} + \dfrac{\partial^2 w_0}{\partial y^2} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} + 2\dfrac{\partial^2 w_0}{\partial x \partial y} \end{Bmatrix},$$

$$\begin{Bmatrix} \gamma_{yz}^{(0)} \\ \gamma_{zx}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \phi_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix}, \quad \begin{Bmatrix} \gamma_{yz}^{(2)} \\ \gamma_{zx}^{(2)} \end{Bmatrix} = -c_2 \begin{Bmatrix} \phi_y + \dfrac{\partial w_0}{\partial y} \\ \phi_x + \dfrac{\partial w_0}{\partial x} \end{Bmatrix}, \quad c_2 = 3c_1, \;\; c_1 = \frac{4}{3}h^2. \tag{11}$$

Taking into account the thermal effects, the stress–strain relationship is as follows:

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{yz} \\ \sigma_{zx} \\ \sigma_{xy} \end{Bmatrix} = \begin{Bmatrix} Q_{11} & Q_{12} & 0 & 0 & 0 \\ Q_{21} & Q_{22} & 0 & 0 & 0 \\ 0 & 0 & Q_{44} & 0 & 0 \\ 0 & 0 & 0 & Q_{55} & 0 \\ 0 & 0 & 0 & 0 & Q_{66} \end{Bmatrix} \left\{ \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{yz} \\ \gamma_{zx} \\ \gamma_{xy} \end{Bmatrix} - \begin{Bmatrix} \alpha_{xx} \\ \alpha_{yy} \\ 0 \\ 0 \\ 2\alpha_{xy} \end{Bmatrix} \Delta T \right\}, \tag{12}$$

where the elastic stiffness coefficients of the FGM plate are given by

$$Q_{11} = Q_{22} = \frac{E}{1 - v^2}, \quad Q_{12} = Q_{21} = \frac{vE}{1 - v^2}, \quad Q_{44} = Q_{55} = Q_{66} = \frac{E}{2(1 + v)} \tag{13}$$

and the coefficient of thermal expansion can be written as

$$\alpha_{xx} = \alpha_{yy} = \alpha, \;\; \alpha_{xy} = 0. \tag{14}$$

According to the Hamilton's principle, the nonlinear governing equations of motion for the FGM rectangular plate are given as

$$N_{xx,x} + N_{xy,y} = I_0 \ddot{u}_0 + (I_1 - c_1 I_3)\ddot{\phi}_x - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial x}, \tag{15a}$$

$$N_{yy,y} + N_{xy,x} = I_0 \ddot{v}_0 + (I_1 - c_1 I_3)\ddot{\phi}_y - c_1 I_3 \frac{\partial \ddot{w}_0}{\partial y}, \tag{15b}$$

$$N_{yy,y}\frac{\partial w_0}{\partial y}+N_{yy}\frac{\partial^2 w_0}{\partial y^2}+N_{xy,x}\frac{\partial w_0}{\partial y}+N_{xy,y}\frac{\partial w_0}{\partial x}+2N_{xy}\frac{\partial^2 w_0}{\partial y\,\partial x}+N_{xx,x}\frac{\partial w_0}{\partial x}+N_{xx}\frac{\partial^2 w_0}{\partial x^2}$$
$$+c_1(P_{xx,xx}+2P_{xy,xy}+P_{yy,yy})+(Q_{x,x}-c_2R_{x,x})+(Q_{y,y}-c_2R_{y,y})+F-\gamma\dot{w}_0$$
$$=I_0\ddot{w}_0+c_1I_3\left(\frac{\partial\ddot{u}_0}{\partial x}+\frac{\partial\ddot{v}_0}{\partial x}\right)+c_1(I_4-c_1I_6)\left(\frac{\partial\ddot{\phi}_x}{\partial x}+\frac{\partial\ddot{\phi}_y}{\partial y}\right)-c_1^2I_6\left(\frac{\partial^2\ddot{w}_0}{\partial x^2}+\frac{\partial^2\ddot{w}_0}{\partial y^2}\right), \quad (15c)$$

$$M_{xx,x}+M_{xy,y}-c_1P_{xx,x}-c_1P_{xy,y}-(Q_x-c_2R_x)=(I_1-c_1I_3)\ddot{u}_0+(I_2-2c_1I_4+c_1^2I_6)\ddot{\phi}_x$$
$$-c_1(I_4-c_1I_6)\frac{\partial\ddot{w}_0}{\partial x}, \quad (15d)$$

$$M_{yy,y}+M_{xy,x}-c_1P_{yy,y}-c_1P_{xy,x}-(Q_y-c_2R_y)=(I_1-c_1I_3)\ddot{v}_0+(I_2-2c_1I_4+c_1^2I_6)\ddot{\phi}_y$$
$$-c_1(I_4-c_1I_6)\frac{\partial\ddot{w}_0}{\partial y}, \quad (15e)$$

where γ is the damping coefficient, a comma denotes the partial differentiation with respect to a specified coordinate, a super dot implies the partial differentiation with respect to time, the stress resultants are represented as follows:

$$\begin{Bmatrix}N_{xx}\\N_{yy}\\N_{xy}\end{Bmatrix}=\{[A][B][E]\}\begin{Bmatrix}\varepsilon^{(0)}\\\varepsilon^{(1)}\\\varepsilon^{(3)}\end{Bmatrix}+\begin{Bmatrix}N_{xx}^{\mathrm{T}}\\N_{yy}^{\mathrm{T}}\\N_{xy}^{\mathrm{T}}\end{Bmatrix},\quad \begin{Bmatrix}M_{xx}\\M_{yy}\\M_{xy}\end{Bmatrix}=\{[B][D][F]\}\begin{Bmatrix}\varepsilon^{(0)}\\\varepsilon^{(1)}\\\varepsilon^{(3)}\end{Bmatrix}+\begin{Bmatrix}M_{xx}^{\mathrm{T}}\\M_{yy}^{\mathrm{T}}\\M_{xy}^{\mathrm{T}}\end{Bmatrix},$$
$$\begin{Bmatrix}P_{xx}\\P_{yy}\\P_{xy}\end{Bmatrix}=\{[E][F][H]\}\begin{Bmatrix}\varepsilon^{(0)}\\\varepsilon^{(1)}\\\varepsilon^{(3)}\end{Bmatrix}+\begin{Bmatrix}P_{xx}^{\mathrm{T}}\\P_{yy}^{\mathrm{T}}\\P_{xy}^{\mathrm{T}}\end{Bmatrix},\quad \begin{Bmatrix}Q_y\\Q_x\end{Bmatrix}=\{[A][D]\}\begin{Bmatrix}\gamma^{(0)}\\\gamma^{(2)}\end{Bmatrix},\quad \begin{Bmatrix}R_y\\R_x\end{Bmatrix}=\{[D][F]\}\begin{Bmatrix}\gamma^{(0)}\\\gamma^{(2)}\end{Bmatrix}, \quad (16)$$

where A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij} and H_{ij}, respectively, are the stiffness elements of the FGM plate, which are denoted as

$$(A_{ij},B_{ij},D_{ij},E_{ij},F_{ij},H_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z,z^2,z^3,z^4,z^6)\,\mathrm{d}z \quad (i,j=1,2,6), \quad (17)$$

$$(A_{ij},D_{ij},F_{ij})=\int_{-h/2}^{h/2}Q_{ij}(1,z^2,z^4)\,\mathrm{d}z \quad (i,j=4,5) \quad (18)$$

and the thermal stress resultants in Eq. (16) can be represented as

$$\begin{Bmatrix}N_{xx}^{\mathrm{T}}\\N_{yy}^{\mathrm{T}}\\N_{xy}^{\mathrm{T}}\end{Bmatrix}=-\int_{-h/2}^{h/2}\begin{bmatrix}Q_{11}&Q_{12}&0\\Q_{21}&Q_{22}&0\\0&0&Q_{66}\end{bmatrix}\begin{Bmatrix}\alpha\\\alpha\\0\end{Bmatrix}\Delta T\,\mathrm{d}z,$$
$$\begin{Bmatrix}M_{xx}^{\mathrm{T}}\\M_{yy}^{\mathrm{T}}\\M_{xy}^{\mathrm{T}}\end{Bmatrix}=-\int_{-h/2}^{h/2}\begin{bmatrix}Q_{11}&Q_{12}&0\\Q_{21}&Q_{22}&0\\0&0&Q_{66}\end{bmatrix}\begin{Bmatrix}\alpha\\\alpha\\0\end{Bmatrix}z\Delta T\,\mathrm{d}z,$$

$$\begin{Bmatrix} P_{xx}^{\mathrm{T}} \\ P_{yy}^{\mathrm{T}} \\ P_{xy}^{\mathrm{T}} \end{Bmatrix} = -\int_{-h/2}^{h/2} \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{21} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix} \begin{Bmatrix} \alpha \\ \alpha \\ 0 \end{Bmatrix} z^3 \Delta T \, \mathrm{d}z. \tag{19}$$

It is found that the N_{xx}^{T} and N_{yy}^{T} in Eq. (19) are the functions with respect to α and ΔT. They represent the thermal stress resultants, which demonstrate the thermal effect.

Substituting the stress resultants of Eq. (16) into Eq. (15), we can write Eq. (15) in terms of generalized displacements (u_0, v_0, w_0, ϕ_x, ϕ_y) as

$$\begin{aligned} & A_{11}\frac{\partial^2 u_0}{\partial x^2} + A_{66}\frac{\partial^2 u_0}{\partial y^2} + (A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\,\partial y} + (B_{11}+c_1E_{11})\frac{\partial^2 \phi_x}{\partial x^2} + (B_{66}+c_1E_{66})\frac{\partial^2 \phi_x}{\partial y^2} \\ & + (B_{12}-c_1E_{12}+B_{66}-c_1E_{66})\frac{\partial^2 \phi_y}{\partial x\,\partial y} + A_{11}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + A_{66}\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} + (A_{12}+A_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\,\partial y} \\ & - c_1E_{11}\frac{\partial^3 w_0}{\partial x^3} - c_1(E_{12}+2E_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2} = I_0\ddot{u}_0 + (I_1 - c_1I_3)\ddot{\phi}_x - c_1I_3\frac{\partial \ddot{w}_0}{\partial x}, \end{aligned} \tag{20a}$$

$$\begin{aligned} & A_{66}\frac{\partial^2 v_0}{\partial x^2} + A_{22}\frac{\partial^2 v_0}{\partial y^2} + (A_{21}+A_{66})\frac{\partial^2 u_0}{\partial x\,\partial y} + (B_{66}+c_1E_{66})\frac{\partial^2 \phi_y}{\partial x^2} + (B_{22}+c_1E_{22})\frac{\partial^2 \phi_y}{\partial y^2} \\ & + (B_{12}-c_1E_{21}+B_{66}-c_1E_{66})\frac{\partial^2 \phi_x}{\partial x\,\partial y} + A_{66}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2} + A_{22}\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2} + (A_{21}+A_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial y} \\ & - c_1E_{22}\frac{\partial^3 w_0}{\partial y^3} - c_1(E_{12}+2E_{66})\frac{\partial^3 w_0}{\partial x^2\,\partial y} = I_0\ddot{v}_0 + (I_1 - c_1I_3)\ddot{\phi}_y - c_1I_3\frac{\partial \ddot{w}_0}{\partial y}, \end{aligned} \tag{20b}$$

$$\begin{aligned} & A_{21}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2} + A_{11}\frac{\partial u_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2} + 2A_{66}\frac{\partial u_0}{\partial y}\frac{\partial^2 w_0}{\partial x\,\partial y} + (A_{21}+A_{66})\frac{\partial^2 u_0}{\partial x\,\partial y}\frac{\partial w_0}{\partial y} + A_{11}\frac{\partial^2 u_0}{\partial x^2}\frac{\partial w_0}{\partial x} + A_{66}\frac{\partial^2 u_0}{\partial y^2}\frac{\partial w_0}{\partial x} \\ & + c_1E_{11}\frac{\partial^3 u_0}{\partial x^3} + c_1(E_{21}+2E_{66})\frac{\partial^3 u_0}{\partial x\,\partial y^2} + 2A_{66}\frac{\partial v_0}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial y} + A_{22}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2} + A_{12}\frac{\partial v_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2} + A_{22}\frac{\partial^2 v_0}{\partial y^2}\frac{\partial w_0}{\partial y} \\ & + A_{66}\frac{\partial^2 v_0}{\partial x^2}\frac{\partial w_0}{\partial y} + (A_{12}+A_{66})\frac{\partial^2 v_0}{\partial x\,\partial y}\frac{\partial w_0}{\partial x} + c_1E_{22}\frac{\partial^3 v_0}{\partial y^3} + c_1(E_{12}+2E_{66})\frac{\partial^3 v_0}{\partial y\,\partial x^2} + (A_{55}+c_2^2F_{55}-2c_2D_{55})\frac{\partial^2 w_0}{\partial x^2} \\ & + (A_{44}+c_2^2F_{44}-2c_2D_{44})\frac{\partial^2 w_0}{\partial y^2} + 2c_1(E_{66}-E_{12})\frac{\partial^2 w_0}{\partial x^2}\frac{\partial^2 w_0}{\partial y^2} + 2c_1E_{66}\frac{\partial^2 w_0}{\partial x\,\partial y}\frac{\partial^2 w_0}{\partial y^2} + 2c_1(E_{21}-2E_{66})\left(\frac{\partial^2 w_0}{\partial x\,\partial y}\right)^2 \\ & + 2(A_{21}+2A_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial y}\frac{\partial w_0}{\partial y} + \frac{3}{2}A_{22}\left(\frac{\partial w_0}{\partial y}\right)^2\frac{\partial^2 w_0}{\partial y^2} + \left(\frac{1}{2}A_{21}+A_{66}\right)\left(\frac{\partial w_0}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial y^2} + \left(A_{66}+\frac{1}{2}A_{21}\right)\left(\frac{\partial w_0}{\partial y}\right)^2\frac{\partial^2 w_0}{\partial x^2} \\ & + \frac{3}{2}A_{11}\left(\frac{\partial w_0}{\partial x}\right)^2\frac{\partial^2 w_0}{\partial x^2} - c_1^2H_{11}\frac{\partial^4 w_0}{\partial x^4} - c_1^2H_{22}\frac{\partial^4 w_0}{\partial y^4} - 2c_1^2(H_{21}-2H_{66})\frac{\partial^4 w_0}{\partial x^2\,\partial y^2} + (-2c_2D_{55}+c_2^2F_{55}+A_{55})\frac{\partial \phi_x}{\partial x} \\ & + (B_{21}-c_1E_{21})\frac{\partial \phi_x}{\partial x}\frac{\partial^2 w_0}{\partial y^2} + 2(B_{66}-c_1E_{66})\frac{\partial \phi_x}{\partial y}\frac{\partial^2 w_0}{\partial x\,\partial y} \\ & + (B_{21}+B_{66}-c_1E_{21}-c_1E_{66})\frac{\partial^2 \phi_x}{\partial x\,\partial y}\frac{\partial w_0}{\partial y} + (B_{11}-c_1E_{11})\frac{\partial^2 \phi_x}{\partial x^2}\frac{\partial w_0}{\partial x} + (B_{11}-c_1E_{11})\frac{\partial \phi_x}{\partial x}\frac{\partial^2 w_0}{\partial x^2} \\ & + (B_{66}-c_1E_{66})\frac{\partial^2 \phi_x}{\partial y^2}\frac{\partial w_0}{\partial x} + (c_1F_{11}-c_1^2H_{11})\frac{\partial^3 \phi_x}{\partial x^3} + c_1(F_{21}+2F_{66}-c_1H_{21}-2c_1H_{66})\frac{\partial^3 \phi_x}{\partial x\,\partial y^2} \\ & + (-2c_2D_{44}+A_{44}+c_2^2F_{44})\frac{\partial \phi_y}{\partial y} + (B_{22}-c_1E_{22})\frac{\partial \phi_y}{\partial y}\frac{\partial^2 w_0}{\partial y^2} \end{aligned}$$

$$+2(B_{66}-c_1E_{66})\frac{\partial \phi_y}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial y}+(B_{66}-c_1E_{66})\frac{\partial^2 \phi_y}{\partial x^2}\frac{\partial w_0}{\partial y}+(B_{12}-c_1E_{12})\frac{\partial \phi_y}{\partial y}\frac{\partial^2 w_0}{\partial x^2}$$
$$+(B_{22}-c_1E_{22})\frac{\partial^2 \phi_y}{\partial y^2}\frac{\partial w_0}{\partial y}+(B_{12}+B_{66}-c_1E_{12}-c_1E_{66})\frac{\partial^2 \phi_y}{\partial x\,\partial y}\frac{\partial w_0}{\partial x}+(c_1F_{22}-c_1^2H_{22})\frac{\partial^3 \phi_y}{\partial y^3}$$
$$+c_1(F_{12}+2F_{66}-c_1H_{12}-2c_1H_{66})\frac{\partial^3 \phi_y}{\partial x^2\,\partial y}-N_{xx}^T\frac{\partial^2 w_0}{\partial x^2}-N_{yy}^T\frac{\partial^2 w_0}{\partial y^2}+F\cos\Omega_1 t$$
$$-\gamma\frac{\partial w_0}{\partial t}=I_0\ddot{w}_0-c_1^2I_6\left(\frac{\partial^2 \ddot{w}_0}{\partial x^2}+\frac{\partial^2 \ddot{w}_0}{\partial y^2}\right)+c_1I_3\left(\frac{\partial \ddot{u}_0}{\partial x}+\frac{\partial \ddot{v}_0}{\partial y}\right)+c_1(I_4-c_1I_6)\left(\frac{\partial \ddot{\phi}_x}{\partial x}+\frac{\partial \ddot{\phi}_y}{\partial y}\right), \tag{20c}$$

$$(B_{11}-c_1E_{11})\frac{\partial^2 u_0}{\partial x^2}+(B_{66}-c_1E_{66})\frac{\partial^2 u_0}{\partial y^2}+(B_{12}+B_{66}-c_1E_{12}-c_1E_{66})\frac{\partial^2 v_0}{\partial x\,\partial y}+(B_{11}-c_1E_{11})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x^2}$$
$$+(B_{12}+B_{66}-c_1E_{12}-c_1E_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x\,\partial y}+(B_{66}-c_1E_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial y^2}+(-c_1F_{11}+c_1^2H_{11})\frac{\partial^3 w_0}{\partial x^3}$$
$$+c_1(-F_{12}-2F_{66}+c_1H_{12}+2c_1H_{66})\frac{\partial^3 w_0}{\partial x\,\partial y^2}-(A_{55}-2c_2D_{55}+c_2^2F_{55})\frac{\partial w_0}{\partial x}+(D_{11}-2c_1F_{11}+c_1^2H_{11})\frac{\partial^2 \phi_x}{\partial x^2}$$
$$+(D_{66}-2c_1F_{66}+c_1^2H_{66})\frac{\partial^2 \phi_x}{\partial y^2}+(D_{12}-2c_1F_{12}-2c_1F_{66}+D_{66}+c_1^2H_{12}+c_1^2H_{66})\frac{\partial^2 \phi_y}{\partial x\,\partial y}$$
$$-(A_{55}-2c_2D_{55}+c_2^2F_{55})\phi_x=(I_1-c_1I_3)\ddot{u}_0+(I_2-2c_1I_4+c_1^2I_6)\ddot{\phi}_x-c_1(I_4-c_1I_6)\frac{\partial \ddot{w}_0}{\partial x}, \tag{20d}$$

$$(B_{66}-c_1E_{66})\frac{\partial^2 v_0}{\partial x^2}+(B_{22}-c_1E_{22})\frac{\partial^2 v_0}{\partial y^2}+(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})\frac{\partial^2 u_0}{\partial x\,\partial y}+(B_{66}-c_1E_{66})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial x^2}$$
$$+(B_{22}-c_1E_{22})\frac{\partial w_0}{\partial y}\frac{\partial^2 w_0}{\partial y^2}+(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})\frac{\partial w_0}{\partial x}\frac{\partial^2 w_0}{\partial x\,\partial y}+(-c_1F_{22}+c_1^2H_{22})\frac{\partial^3 w_0}{\partial y^3}$$
$$+c_1(-F_{21}-2F_{66}+c_1H_{21}+2c_1H_{66})\frac{\partial^3 w_0}{\partial x^2\,\partial y}-(A_{44}-2c_2D_{44}+c_2^2F_{44})\frac{\partial w_0}{\partial y}+(D_{66}-2c_1F_{66}+c_1^2H_{66})\frac{\partial^2 \phi_y}{\partial x^2}$$
$$+(D_{22}-2c_1F_{22}+c_1^2H_{22})\frac{\partial^2 \phi_y}{\partial y^2}+(D_{21}-2c_1F_{21}-2c_1F_{66}+D_{66}+c_1^2H_{21}+c_1^2H_{66})\frac{\partial^2 \phi_x}{\partial x\,\partial y}$$
$$-(A_{44}-2c_2D_{44}+c_2^2F_{44})\phi_y=(I_1-c_1I_3)\ddot{v}_0+(I_2-2c_1I_4+c_1^2I_6)\ddot{\phi}_y-c_1(I_4-c_1I_6)\frac{\partial \ddot{w}_0}{\partial y}, \tag{20e}$$

where all kinds of inertias in Eq. (20) are calculated by

$$I_i=\int_{-h/2}^{h/2} z^i\rho(z)\,\mathrm{d}z \quad (i=0,\ 1,\ 2,\ 3,\ 4,\ 6). \tag{21}$$

The simply supported boundary conditions can be expressed as

$$\text{at } x=0 \text{ and } x=a,\ \ w=\phi_y=M_{xx}=P_{xx}=N_{xy}=0, \tag{22a}$$

$$\text{at } y=0 \text{ and } y=b,\ \ w=\phi_x=M_{yy}=P_{yy}=N_{xy}=0, \tag{22b}$$

$$N_{yy}|_{y=0,\,b}=0,\quad \int_0^b N_{xx}|_{x=0,a}\,\mathrm{d}y=-\int_0^b (p_0-p_1\cos\Omega_2 t)\,\mathrm{d}y. \tag{22c}$$

It is obvious that the boundary condition (22c) also includes the influence of the thermal environment.

In order to obtain the dimensionless equations, we introduce the transformations of the variables and parameters

$$\bar{u}=\frac{u_0}{a},\quad \bar{v}=\frac{v_0}{b},\quad \bar{w}=\frac{w_0}{h},\quad \bar{\phi}_x=\phi_x,\quad \bar{\phi}_y=\phi_y,\quad \bar{x}=\frac{x}{a},\quad \bar{y}=\frac{y}{b},\quad \bar{F}=\frac{(ab)^{7/2}}{\pi^4 E h^7}F,$$
$$\bar{\gamma}=\frac{(ab)^2}{\pi^2 h^4}\left(\frac{1}{\rho E}\right)^{1/2}\gamma,\quad \bar{P}_0=\frac{b^2}{Eh^3}p_0,\quad \bar{P}_1=\frac{b^2}{Eh^3}p_1,\quad \bar{t}=\pi^2\left(\frac{E}{ab\rho}\right)^{1/2}t,$$
$$\bar{\Omega}_i=\frac{1}{\pi^2}\left(\frac{ab\rho}{E}\right)^{1/2}\Omega_i\ (i=1,2),\quad \bar{A}_{ij}=\frac{(ab)^{1/2}}{Eh^2}A_{ij},\quad \bar{B}_{ij}=\frac{(ab)^{1/2}}{Eh^3}B_{ij},\quad \bar{D}_{ij}=\frac{(ab)^{1/2}}{Eh^4}D_{ij},$$
$$\bar{E}_{ij}=\frac{(ab)^{1/2}}{Eh^5}E_{ij},\quad \bar{F}_{ij}=\frac{(ab)^{1/2}}{Eh^6}F_{ij},\quad \bar{H}_{ij}=\frac{(ab)^{1/2}}{Eh^8}H_{ij},\quad \bar{I}_i=\frac{1}{(ab)^{(i+1)/2}\rho}I_i. \tag{23}$$

We mainly consider transverse nonlinear oscillations of the FGM rectangular plate in the first two modes. It is our aim to choose a suitable mode function to satisfy the first two modes of transverse nonlinear oscillations and the boundary conditions for the FGM rectangular plates. Thus, we write $\bar{w}$ as follows:

$$\bar{w}(\bar{x},\bar{y},\bar{t})=\bar{w}_1(\bar{t})\sin\frac{\pi\bar{x}}{a}\sin\frac{3\pi\bar{y}}{b}+\bar{w}_2(\bar{t})\sin\frac{3\pi\bar{x}}{a}\sin\frac{\pi\bar{y}}{b}, \tag{24}$$

where $\bar{w}_1$ and $\bar{w}_2$ are the amplitudes of two modes, respectively.

The transverse excitation can be represented as

$$\bar{F}(\bar{x},\bar{y},\bar{t})=\bar{F}_1(\bar{t})\sin\frac{\pi\bar{x}}{a}\sin\frac{3\pi\bar{y}}{b}+\bar{F}_2(\bar{t})\sin\frac{3\pi\bar{x}}{a}\sin\frac{\pi\bar{y}}{b}, \tag{25}$$

where $\bar{F}_1$ and $\bar{F}_2$ represent the amplitudes of the transverse forcing excitation corresponding to the two nonlinear modes.

For simplicity, we drop the overbar in the following analysis. Based on research given in Refs. [33,34], neglecting all inertia terms on u, v, ϕ_x and ϕ_y in Eq. (20) and substituting Eq. (24) into Eqs. (20a), (20b), (20d) and (20e), we obtain the displacements u, v, ϕ_x and ϕ_y with respect to w. Substituting Eqs. (24) and (25) into Eq. (20c) and applying the Galerkin procedure yield the governing differential equation of transverse motion of the FGM rectangular plate for the dimensionless as follows:

$$\ddot{w}_1+\omega_1^2 w_1+a_1\dot{w}_1+a_2 w_1\cos\Omega_2 t+a_3 w_1^2+a_4 w_2^2+a_5 w_1 w_2^2+a_6 w_1^3+a_7 w_1 w_2=f_1\cos\Omega_1 t, \tag{26a}$$

$$\ddot{w}_2+\omega_2^2 w_2+b_1\dot{w}_2+b_2 w_2\cos\Omega_2 t+b_3 w_1 w_2+b_4 w_1^2+b_5 w_2^2+b_6 w_2 w_1^2+b_7 w_2^3=f_2\cos\Omega_1 t, \tag{26b}$$

where all coefficients can be found in Appendix A, the f_1 and f_2 are the magnitudes of the forcing excitations, which are also given in Appendix A.

The aforementioned equation, which includes the quadratic terms, cubic terms, parametric and transverse excitations, describes the nonlinear transverse vibrations of the FGM rectangular plate subjected to the in-plane and transversal excitations in the first two modes. To consider the influence of the quadratic terms on the nonlinear dynamic characteristics of the FGM rectangular plate, we need to obtain the second-order approximate solution of Eq. (26). It is difficult for one to use the method of multiple scales to obtain the second-order approximate solution of Eq. (26). Thus, in the following analysis, we will utilize the asymptotic perturbation method presented by Maccari [21–25] to obtain the second-order approximate solution of Eq. (26). Utilizing the asymptotic perturbation method, we can obtain increasingly accurate solution by increasing the order of approximation in terms of the small parameter ε.

3. Perturbation analysis

To guarantee the validity of perturbation analysis, we use the asymptotic perturbation method [21–26,35] to obtain the averaged equation of system (26).

It is assumed that the width-to-length ratio of the FGM rectangular plate is $a/b=1$. Therefore, we only consider the case of 1:1 internal resonance and primary parametric resonance for the FGM rectangular plate.

In this resonant case, there are the following relations:

$$\omega_1 = \frac{\Omega_1}{2} + \varepsilon^2\sigma_1,\ \ \omega_2 = \frac{\Omega_1}{2} + \varepsilon^2\sigma_2,\ \ \Omega_2 = \Omega_1 = \Omega, \tag{27}$$

where ω_1 and ω_2 are two linear frequencies, σ_1 and σ_2 are the two detuning parameters.

The scale transformations may be introduced as

$$a_1 \to \varepsilon^2 a_1,\ \ a_2 \to \varepsilon^2 a_2,\ \ f_1 \to \varepsilon^2 f_1,\ \ b_1 \to \varepsilon^2 b_1,\ \ b_2 \to \varepsilon^2 b_2,\ \ f_2 \to \varepsilon^2 f_2. \tag{28}$$

Substituting Eqs. (27) and (28) into Eq. (26) yields

$$\ddot{w}_1 + \left(\frac{\Omega^2}{4} + \varepsilon^4\sigma_1^2 + \Omega\sigma_1\varepsilon^2\right)w_1 + \varepsilon^2 a_1\dot{w}_1 + \varepsilon^2 a_2 w_1 \cos\Omega t + a_3 w_1^2 + a_4 w_2^2 + a_5 w_1 w_2^2 + a_6 w_1^3 + a_7 w_1 w_2$$
$$= \varepsilon^2 f_1 \cos\Omega t, \tag{29a}$$

$$\ddot{w}_2 + \left(\frac{\Omega^2}{4} + \varepsilon^4\sigma_2^2 + \Omega\sigma_2\varepsilon^2\right)w_2 + \varepsilon^2 b_1\dot{w}_2 + \varepsilon^2 b_2 w_2 \cos\Omega t + b_3 w_1 w_2 + b_4 w_1^2 + b_5 w_2^2 + b_6 w_2 w_1^2 + b_7 w_2^3$$
$$= \varepsilon^2 f_2 \cos\Omega t, \tag{29b}$$

We now introduce the temporal rescaling

$$\tau = \varepsilon^q t, \tag{30}$$

where q is a rational positive number, which will be fixed afterwards.

The value of q fixes the magnitude order of the temporal asymptotic limit in such a way that the nonlinear effects become consistent and non-negligible. If $t \to \infty$, we set $\varepsilon \to 0$, so that τ assumes a finite value.

The approximate solutions $w_1(t)$ and $w_2(t)$ of Eq. (29) are sought in a power series of small parameter ε

$$w_1(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\delta_n}\psi_n(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}n(\Omega/2)t}, \tag{31a}$$

$$w_2(t) = \sum_{n=-\infty}^{+\infty} \varepsilon^{\delta_n}\Phi_n(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}n(\Omega/2)t}, \tag{31b}$$

where $\delta_n = |n|$ for $n \neq 0$, and $\delta_0 = \delta$ is a positive number, which will be fixed later on.

Because $w_1(t)$ and $w_2(t)$ are real, we can obtain

$$\psi_n(\tau,\varepsilon) = \psi^*_{-n}(\tau,\varepsilon), \tag{32a}$$

$$\Phi_n(\tau,\varepsilon) = \Phi^*_{-n}(\tau,\varepsilon), \tag{32b}$$

where the asterisk denotes the complex conjugate.

Therefore, the assumed solution (31) can be rewritten more explicitly as follows:

$$w_1(t) = \varepsilon^\delta\psi_0(\tau,\varepsilon) + \varepsilon\psi_1(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}(\Omega/2)t} + \varepsilon^2\psi_2(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}\Omega t} + \varepsilon^3\psi_3(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}(3\Omega/2)t}$$
$$+ \varepsilon^4\psi_4(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}2\Omega t} + \mathrm{cc} + O(\varepsilon^5), \tag{33a}$$

$$w_2(t) = \varepsilon^\delta\Phi_0(\tau,\varepsilon) + \varepsilon\Phi_1(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}(\Omega/2)t} + \varepsilon^2\Phi_2(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}\Omega t} + \varepsilon^3\Phi_3(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}(3\Omega/2)t}$$
$$+ \varepsilon^4\Phi_4(\tau,\varepsilon)\,\mathrm{e}^{-\mathrm{i}2\Omega t} + \mathrm{cc} + O(\varepsilon^5), \tag{33b}$$

where the symbol cc stands for the parts of complex conjugate of the functions on the right-hand side of Eq. (33).

It is seen from Eq. (33) that the solutions of Eq. (29) can be considered as a combination of the various harmonics with coefficients depending on τ and ε. Assume that the functions $\psi_n(\tau,\varepsilon)$ and $\Phi_n(\tau,\varepsilon)$ can be expanded in power series of small parameter ε

$$\psi_n(\tau,\varepsilon)=\sum_{i=0}^{+\infty}\varepsilon^i\psi_n^{(i)}(\tau), \tag{34a}$$

$$\Phi_n(\tau,\varepsilon)=\sum_{i=0}^{+\infty}\varepsilon^i\Phi_n^{(i)}(\tau). \tag{34b}$$

It is also supposed that the limits of functions $\psi_n(\tau,\varepsilon)$ and $\Phi_n(\tau,\varepsilon)$ as $\varepsilon\to 0$ exist and are finite. For simplicity of analysis, we use abbreviations $\psi_n^{(0)}=\psi_n$ and $\Phi_n^{(0)}=\Phi_n$ for $n\neq 1$ and $\psi_1^{(0)}=\psi$, $\Phi_1^{(0)}=\Phi$ for $n=1$. Note that the introduction of the temporal rescaling (30) implies that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\psi_n\,\mathrm{e}^{-\mathrm{i}n(\Omega/2)t})=\left(-\mathrm{i}n(\Omega/2)\psi_n+\varepsilon^q\frac{\mathrm{d}\psi_n}{\mathrm{d}\tau}\right)\mathrm{e}^{-\mathrm{i}n(\Omega/2)t}, \tag{35a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Phi_n\,\mathrm{e}^{-\mathrm{i}n(\Omega/2)t})=\left(-\mathrm{i}n(\Omega/2)\Phi_n+\varepsilon^q\frac{\mathrm{d}\Phi_n}{\mathrm{d}\tau}\right)\mathrm{e}^{-\mathrm{i}n(\Omega/2)t}. \tag{35b}$$

In order to determine the coefficients $\psi_n(\tau,\varepsilon)$ and $\Phi_n(\tau,\varepsilon)$, we substitute solution (31) into Eq. (29) and obtain the equations for each harmonic with order n and for a fixed order of approximation on the perturbation parameter ε.

For $n=0$, we obtain

$$\frac{\Omega^2}{4}\varepsilon^\eta\psi_0+2a_3\varepsilon^2|\psi_1|^2+2a_4\varepsilon^2|\Phi_1|^2+a_7\varepsilon^2(\psi_1\Phi_1^*+\psi_1^*\Phi_1)=0, \tag{36a}$$

$$\frac{\Omega^2}{4}\varepsilon^\eta\Phi_0+b_3\varepsilon^2(\psi_1\Phi_1^*+\psi_1^*\Phi_1)+2b_4\varepsilon^2|\psi_1|^2+2b_5\varepsilon^2|\Phi_1|^2=0. \tag{36b}$$

The correct balance of the terms indicates $\delta=2$. Therefore, we derive the following relation:

$$\psi_0=-\frac{8a_3|\psi_1|^2+8a_4|\Phi_1|^2+4a_7(\Phi_1^*\psi_1+\psi_1^*\Phi_1)}{\Omega^2}, \tag{37a}$$

$$\Phi_0=-\frac{4b_3(\Phi_1^*\psi_1+\psi_1^*\Phi_1)+8b_4|\psi_1|^2+8b_5|\Phi_1|^2}{\Omega^2}. \tag{37b}$$

For $n=2$, taking into account Eq. (35) yields

$$\tfrac{3}{4}\Omega^2\psi_2=a_3\psi_1^2+a_4\Phi_1^2-\tfrac{1}{2}f_1+a_7\psi_1\Phi_1, \tag{38a}$$

$$\tfrac{3}{4}\Omega^2\Phi_2=b_3\psi_1\Phi_1+b_4\psi_1^2+b_5\Phi_1^2-\tfrac{1}{2}f_2. \tag{38b}$$

and the corresponding relations

$$\psi_2=\frac{4}{3\Omega^2}\left(a_3\psi_1^2+a_4\Phi_1^2-\frac{1}{2}f_1+a_7\psi_1\Phi_1\right), \tag{39a}$$

$$\Phi_2=\frac{4}{3\Omega^2}\left(b_3\psi_1\Phi_1+b_4\psi_1^2+b_5\Phi_1^2-\frac{1}{2}f_2\right). \tag{39b}$$

From the proper balance of the nonlinear and linear terms, for $n = 1$, we must consider $q = 2$. Based on Eq. (29), we have

$$
\begin{aligned}
&-\Omega \mathrm{i}\frac{\mathrm{d}\psi_1}{\mathrm{d}\tau} + \left(\Omega\sigma_1 - a_1\frac{\Omega}{2}\mathrm{i}\right)\psi_1 + \left(\frac{1}{2}a_2 - \frac{4}{3\Omega^2}a_3 f_1\right)\psi_1^* + \left[\left(-\frac{40}{3\Omega^2}a_3^2 + 3a_6 + \frac{20}{3\Omega^2}a_7 b_4\right)|\psi_1|^2\right.\\
&\left.+\left(-\frac{16}{\Omega^2}a_3 a_4 - \frac{16}{3\Omega^2}a_4 b_3 + 2a_5 - \frac{8}{\Omega^2}a_7 b_5 - \frac{8}{3\Omega^2}a_7^2\right)|\Phi_1|^2\right]\psi_1 + \left(-\frac{16}{\Omega^2}a_4 b_4 + \frac{16}{\Omega^2}a_7 b_3\right.\\
&\left.+\frac{8}{\Omega^2}a_3 a_7\right)|\psi_1|^2\Phi_1 + \left(-\frac{40}{3\Omega^2}a_4 b_5 - \frac{20}{3\Omega^2}a_4 a_7\right)|\Phi_1|^2\Phi_1 + \left(\frac{8}{3\Omega^2}a_3 a_4 - \frac{8}{\Omega^2}a_4 b_3 + a_5\right.\\
&\left.+\frac{4}{\Omega^2}a_7 b_5\right)\Phi_1^2\psi_1^* + \left(\frac{8}{3\Omega^2}a_4 b_4 - \frac{20}{3\Omega^2}a_3 b_7 - \frac{4}{\Omega^2}a_7 b_3\right)\psi_1^2\Phi_1^* + 4a_7\psi_1\Phi_1^2 - \frac{4}{3\Omega^2}a_4 f_2\Phi_1^*\\
&-\frac{2}{3\Omega^2}a_7 f_1\Phi_1^* - \frac{4}{3\Omega^2}a_7 f_2\psi_1^* = 0, \qquad (40a)
\end{aligned}
$$

$$
\begin{aligned}
&-\Omega \mathrm{i}\frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau} + \left(\Omega\sigma_2 - b_1\frac{\Omega}{2}\mathrm{i}\right)\Phi_1 + \left(\frac{1}{2}b_2 - \frac{2}{3\Omega^2}b_3 f_1 - \frac{4}{3\Omega^2}b_5 f_2\right)\Phi_1^*\\
&-\left(\frac{2}{3\Omega^2}b_3 f_2 + \frac{4}{3\Omega^2}b_4 f_1\right)\psi_1^* + \left(-\frac{8}{\Omega^2}a_3 b_3 - \frac{16}{3\Omega^2}b_3^2 - \frac{16}{\Omega^2}b_4 b_5 + 2b_6 + \frac{32}{3\Omega^2}a_7 b_4\right)|\psi_1|^2\Phi_1\\
&+\left(-\frac{20}{\Omega^2}a_4 b_3 - \frac{40}{3\Omega^2}b_5^2 + 3b_7\right)|\Phi_1|^2\Phi_1 + \left(-\frac{20}{3\Omega^2}b_3 b_4 - \frac{40}{3\Omega^2}a_3 b_4\right)|\psi_1|^2\psi_1\\
&+\left(-\frac{8}{3\Omega^2}a_7 b_3 - \frac{16}{\Omega^2}a_4 b_4 - \frac{40}{3\Omega^2}b_3 b_5\right)|\Phi_1|^2\psi_1\\
&+\left(-\frac{4}{\Omega^2}b_3^2 + \frac{4}{3\Omega^2}a_3 b_3 + \frac{8}{3\Omega^2}b_4 b_5 + \frac{8}{\Omega^2}b_4 a_7\right)\psi_1^2\Phi_1^*\\
&+\left(\frac{28}{3\Omega^2}b_3 b_5 + \frac{8}{3\Omega^2}a_4 b_4 + b_6 - \frac{4}{\Omega^2}a_7 b_3\right)\psi_1^*\Phi_1^2 = 0. \qquad (40b)
\end{aligned}
$$

Based on Eqs. (26) and (28), the differential equation for the evolution of the complex amplitudes ψ_1 and Φ_1 can be derived as

$$
\begin{aligned}
\frac{\mathrm{d}\psi_1}{\mathrm{d}\tau} &= \mu_1\psi_1 - \sigma_1\psi_1\mathrm{i} + (\alpha_1 + \alpha_2 f_1 + \alpha_{11} f_2)\psi_1^*\mathrm{i} + (\alpha_3 f_2 + \alpha_{11} f_1)\Phi_1^* + \alpha_4|\psi_1|^2\psi_1\mathrm{i} + \alpha_5|\phi_1|^2\psi_1\mathrm{i}\\
&\quad + \alpha_6|\psi_1|^2\phi_1\mathrm{i} + \alpha_7|\phi_1|^2\phi_1\mathrm{i} + \alpha_8\phi_1^2\psi_1^*\mathrm{i} + \alpha_9\psi_1^2\phi_1^*\mathrm{i} + \alpha_{10}\phi_1^2\psi_1\mathrm{i}, \qquad (41a)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\Phi_1}{\mathrm{d}\tau} &= \mu_2\Phi_1 - \sigma_2\Phi_1\mathrm{i} + (\beta_1 + \beta_2 f_1 + \beta_3 f_2)\Phi_1^*\mathrm{i} + (\beta_4 f_2 + \beta_5 f_1)\psi_1^*\mathrm{i} + \beta_6|\psi_1|^2\Phi_1\mathrm{i} + \beta_7|\Phi_1|^2\Phi_1\mathrm{i}\\
&\quad + \beta_8|\psi_1|^2\psi_1\mathrm{i} + \beta_9|\Phi_1|^2\psi_1\mathrm{i} + \beta_{10}\psi_1^2\Phi_1^*\mathrm{i} + \beta_{11}\psi_1^*\Phi_1^2\mathrm{i}. \qquad (41b)
\end{aligned}
$$

In order to transform Eq. (41) into the Cartesian form, let

$$\psi_1 = x_1 + \mathrm{i}x_2, \quad \Phi_1 = x_3 + \mathrm{i}x_4. \qquad (42)$$

Substituting Eq. (42) into Eq. (41), the averaged equation in the Cartesian form is obtained as follows:

$$
\begin{aligned}
\frac{\mathrm{d}x_1}{\mathrm{d}\tau} &= \mu_1 x_1 + (\sigma_1 + \alpha_1 + \alpha_2 f_1 + \alpha_{11} f_2)x_2 + (\alpha_3 f_2 + \alpha_{11} f_1)x_4 + \alpha_4 x_1^2 x_2 + \alpha_4 x_2^3 + (\alpha_{10} - \alpha_8 + \alpha_5)x_3^2 x_2\\
&\quad + (\alpha_5 + \alpha_8 - \alpha_{10})x_4^2 x_2 + 2\alpha_9 x_1 x_2 x_3 + (-\alpha_9 + \alpha_6)x_1^2 x_4 + (\alpha_6 + \alpha_9)x_2^2 x_4 + 2(\alpha_8 + \alpha_{10})x_1 x_3 x_4\\
&\quad + \alpha_7 x_4^3 + \alpha_7 x_3^2 x_4, \qquad (43a)
\end{aligned}
$$

$$\frac{dx_2}{d\tau} = (-\sigma_1 + \alpha_1 + \alpha_2 f_1 + \alpha_{11} f_2)x_1 + \mu_1 x_2 + (\alpha_3 f_2 + \alpha_{11} f_1)x_3 - \alpha_4 x_1^3 - \alpha_4 x_2^2 x_1 - (\alpha_8 + \alpha_5 + \alpha_{10})x_3^2 x_1 - (\alpha_5 - \alpha_8 - \alpha_{10})x_4^2 x_1 - 2\alpha_9 x_1 x_2 x_4 - (\alpha_9 + \alpha_6)x_1^2 x_3 - (\alpha_6 - \alpha_9)x_2^2 x_3 - 2(\alpha_8 - \alpha_{10})x_2 x_3 x_4 - \alpha_7 x_4^2 x_3 - \alpha_7 x_3^3, \tag{43b}$$

$$\frac{dx_3}{d\tau} = (\beta_4 f_2 + \beta_5 f_1)x_2 + \mu_2 x_3 + (\sigma_2 + \beta_1 + \beta_2 f_1 + \beta_3 f_2)x_4 + \beta_8 x_1^2 x_2 + (\beta_6 - \beta_{10})x_1^2 x_4 + (\beta_6 + \beta_{10})x_2^2 x_4 + \beta_8 x_2^3 - (\beta_{11} - \beta_9)x_3^2 x_2 + \beta_7 x_3^2 x_4 + (\beta_9 + \beta_{11})x_4^2 x_2 + \beta_7 x_4^3 + 2\beta_{10} x_1 x_2 x_3 + 2\beta_{11} x_1 x_3 x_4, \tag{43c}$$

$$\frac{dx_4}{d\tau} = (\beta_4 f_2 + \beta_5 f_1)x_1 + (-\sigma_2 + \beta_1 + \beta_2 f_1 + \beta_3 f_2)x_3 + \mu_2 x_4 - \beta_8 x_1^3 - (\beta_6 + \beta_{10})x_1^2 x_3 + (\beta_6 - \beta_{10})x_2^2 x_3 - \beta_8 x_2^2 x_1 - (\beta_9 + \beta_{11})x_3^2 x_1 - \beta_7 x_3^3 + (\beta_9 - \beta_{11})x_4^2 x_1 - \beta_7 x_4^2 x_3 - 2\beta_{11} x_2 x_3 x_4 - 2\beta_{10} x_1 x_2 x_4, \tag{43d}$$

where all coefficients can be found in Appendix B.

4. Numerical simulations of periodic and chaotic motions

In the following investigation, the fourth-order Runge–Kutta algorithm [36] is utilized to numerically analyze the periodic and chaotic motions of the FGM rectangular plate subjected to thermal and mechanical loads for the case of 1:1 internal resonance and primary parametric resonance. We consider the averaged equation (43) to carry out numerical simulation. We choose the forcing excitations f_1 and f_2 as the controlling parameters when the periodic and chaotic responses of the FGM rectangular plate are investigated. Zirconia and titanium alloy are selected for the two constituent materials of the FGM plate in this example, referred to as ZrO_2/Ti–6Al–4V. The properties of this material can be found in Ref. [37]. The two-dimensional phase portrait, waveform, three-dimensional phase portrait and Poincare map are plotted to demonstrate the nonlinear dynamic behaviors of the FGM rectangular plate. It can be clearly found from the numerical results that the periodic and chaotic motions occur for the FGM rectangular plate.

Fig. 2 illustrates the existence of the chaotic motion for the FGM rectangular plate when the forcing excitation f_2 is 6.99. The parameters and the initial conditions are, respectively, chosen as $\sigma_1 = 0.52$, $\sigma_2 = 0.86$, $\mu_1 = 0.32$, $\mu_2 = 0.32$, $\alpha_1 = 5.2$, $\alpha_2 = 12.2$, $\alpha_3 = -8.1$,$\alpha_4 = 0.66$, $\alpha_5 = -0.5$, $\alpha_6 = 4.21$, $\alpha_7 = -0.36$, $\alpha_8 = 3.3$, $\alpha_9 = 12.2$, $\alpha_{10} = 3.26$, $\alpha_{11} = -5.78$, $\beta_1 = 4.4$, $\beta_2 = 15.2$, $\beta_3 = -8.73$, $\beta_4 = 3.03$, $\beta_5 = -5.1$, $\beta_6 = 2.2$, $\beta_7 = -6.98$, $\beta_8 = -11.3$, $\beta_9 = 5.21$, $\beta_{10} = -4.15$, $\beta_{11} = 0.66$, $f_1 = 8.62$, $x_{10} = 0.44$, $x_{20} = 1.55$, $x_{30} = 2.35$, $x_{40} = 5.18$. Figs. 2(a) and (c) represent the phase portraits on the planes (x_1,x_2) and (x_3,x_4), respectively. Figs. 2(b) and (d) respectively denote the waveforms on the planes (t,x_1) and (t,x_3). Figs. 2(e) and (f) represent the three-dimensional phase portrait in space (x_1, x_2, x_3) and the Poincare map on plane (x_1, x_2), respectively. It can be shown from Fig. 2 that the amplitude of the second-order mode is larger than one of the first-order mode. Until the forcing excitation is increased to $f_2 = 7.008$, the response of the FGM rectangular plate also is the chaotic motion, as shown in Fig. 3. The Poincare maps given in Figs. 2(f)–3(f) clearly demonstrate that chaotic motions exist for the FGM rectangular plate.

Fig. 4 indicates that the quasi-period motion of the FGM rectangular plate occurs when the forcing excitation changes to $f_2 = 7.012$. Fig. 5 shows that the chaotic motions of the FGM rectangular plate again occur when the forcing excitation changes to $f_2 = 7.038$. Fig. 6 illustrates that the quasi-period response of the FGM rectangular plate occurs when $f_2 = 7.053$. When $f_2 = 7.098$, the chaotic response of the FGM rectangular plate exists for the FGM rectangular plate, as shown in Fig. 7.

Continuously increasing the forcing excitation to $f_2 = 7.288$, it is found that period-9 solution occurs for the FGM rectangular plate, as shown in Fig. 8. In Fig. 9, it is seen that the multiple period motion of the FGM rectangular plate exists when $f_2 = 7.389$. With the increasing of the forcing excitation continuously, the periodic motion of the FGM rectangular plate also exists when $f_2 = 7.403$, as shown in Fig. 10. Because of limited space, we do not provide other figures.

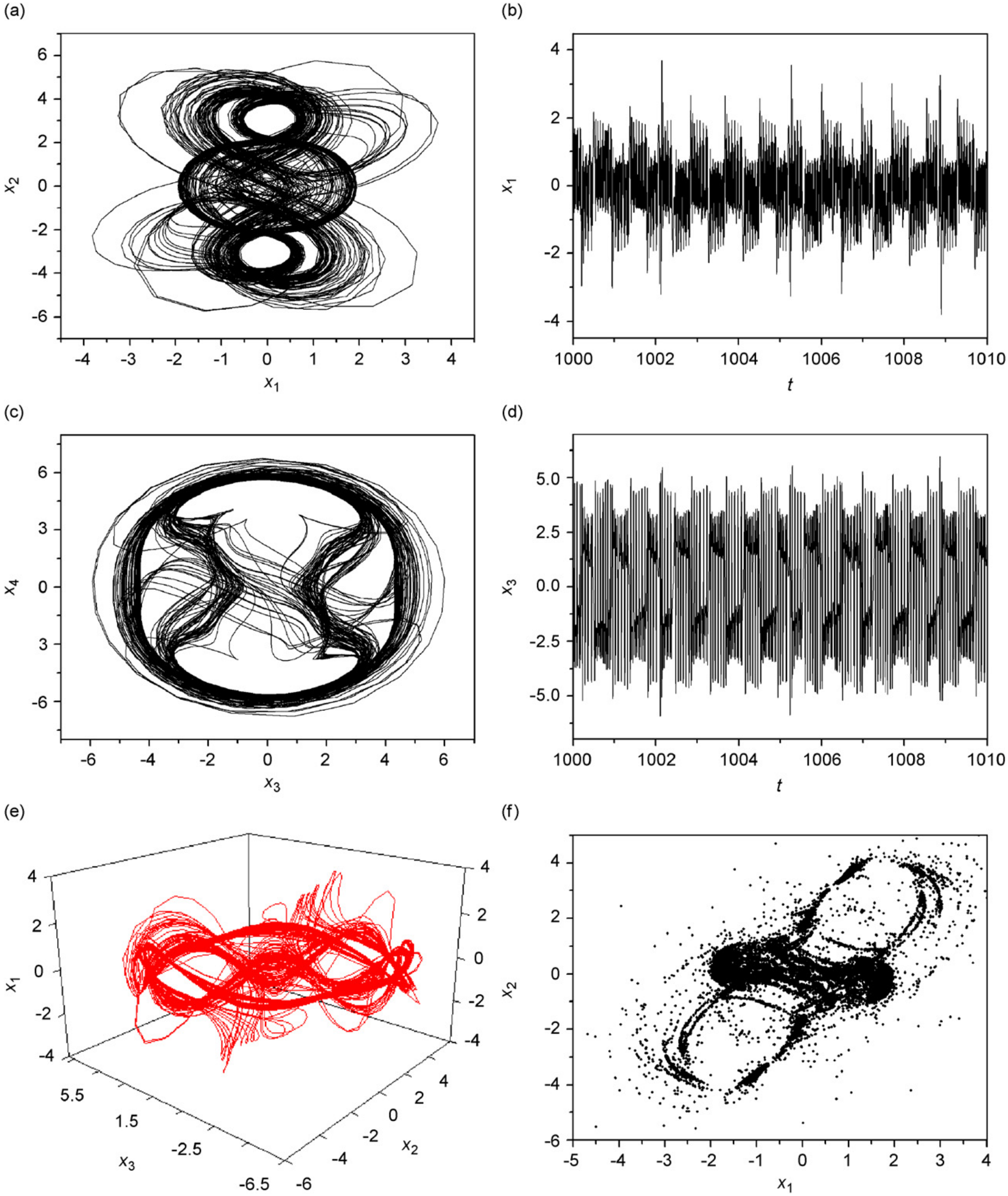

Fig. 2. The chaotic motion of the FGMs rectangular plate exists when $f_2 = 6.99$, (a) the phase portrait on plane (x_1, x_2); (b) the waveforms on the planes (t, x_1); (c) the phase portrait on plane (x_3, x_4); (d) the waveforms on the planes (t, x_3); (e) three-dimensional phase portrait in space (x_1, x_2, x_3); and (f) the Poincare map on plane (x_1, x_2).

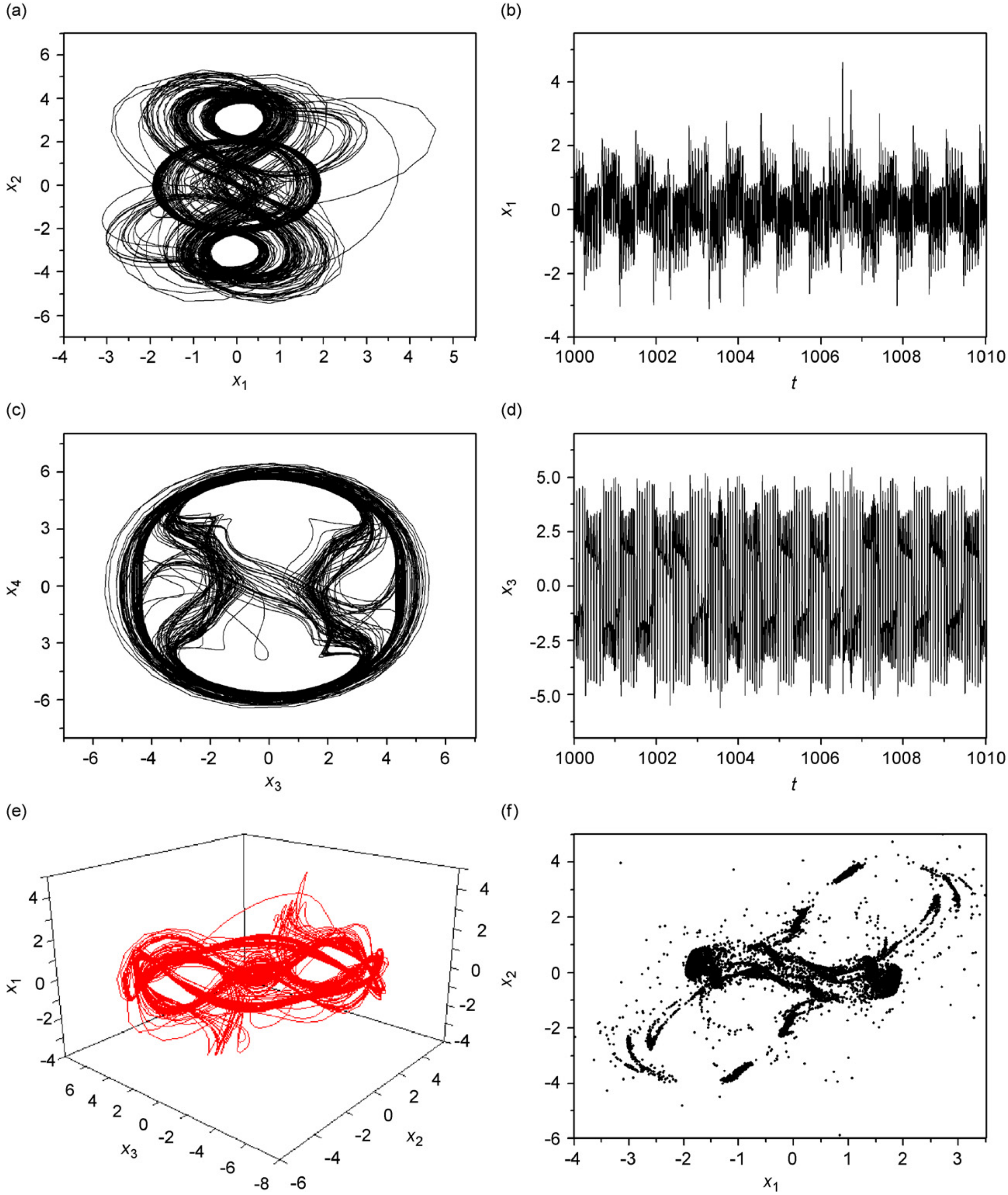

Fig. 3. The chaotic motion of the FGMs rectangular plate exists when $f_2 = 7.008$.

From Figs. 2–10, it can be shown that the process of change for the motions of the FGM rectangular plate is as follows: the chaotic motion → the quasi-period motion → the chaotic motion → the quasi-period motion → the chaotic motion → the period n motion.

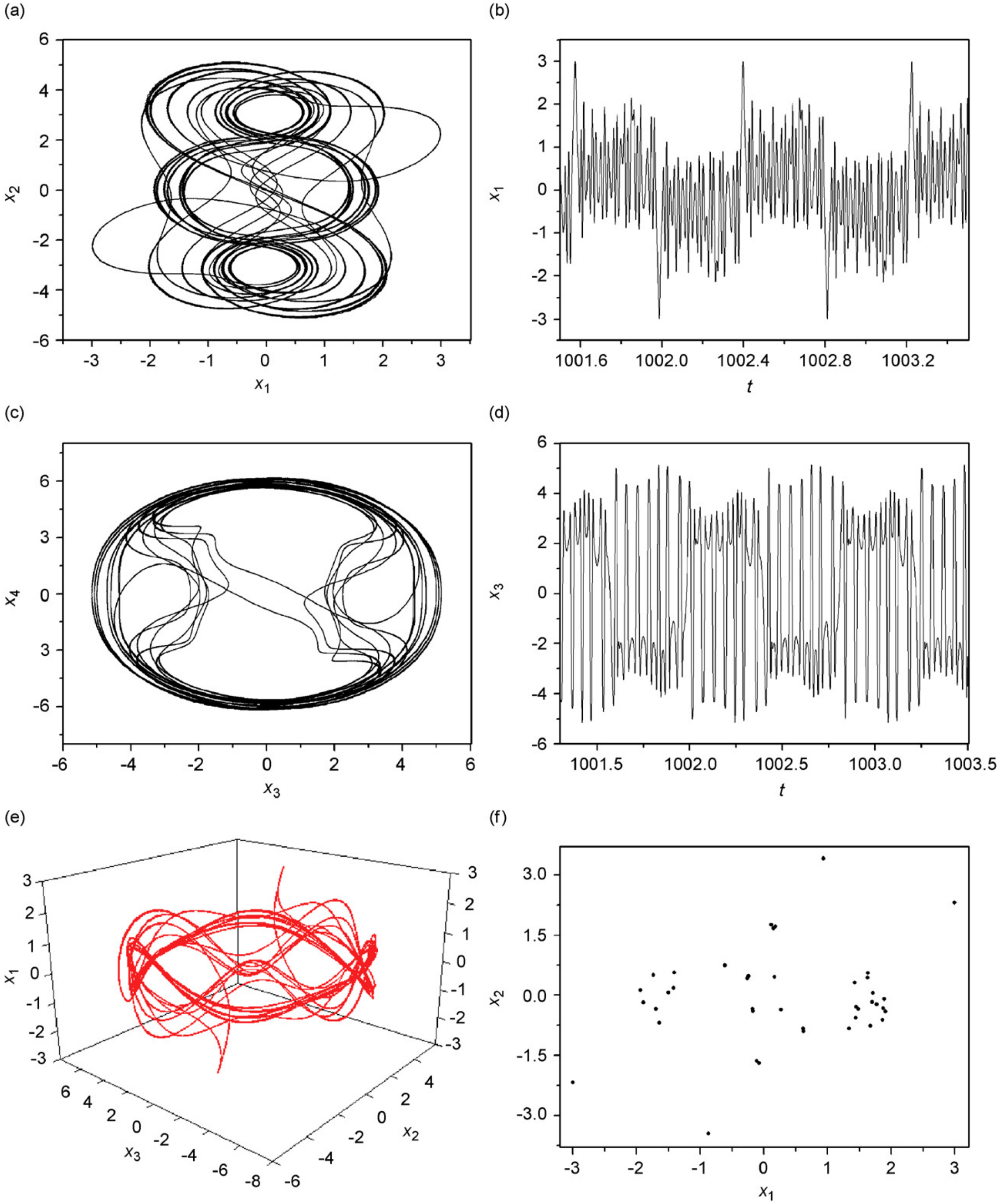

Fig. 4. The quasi-period motion of the FGMs rectangular plate exists when $f_2 = 7.012$.

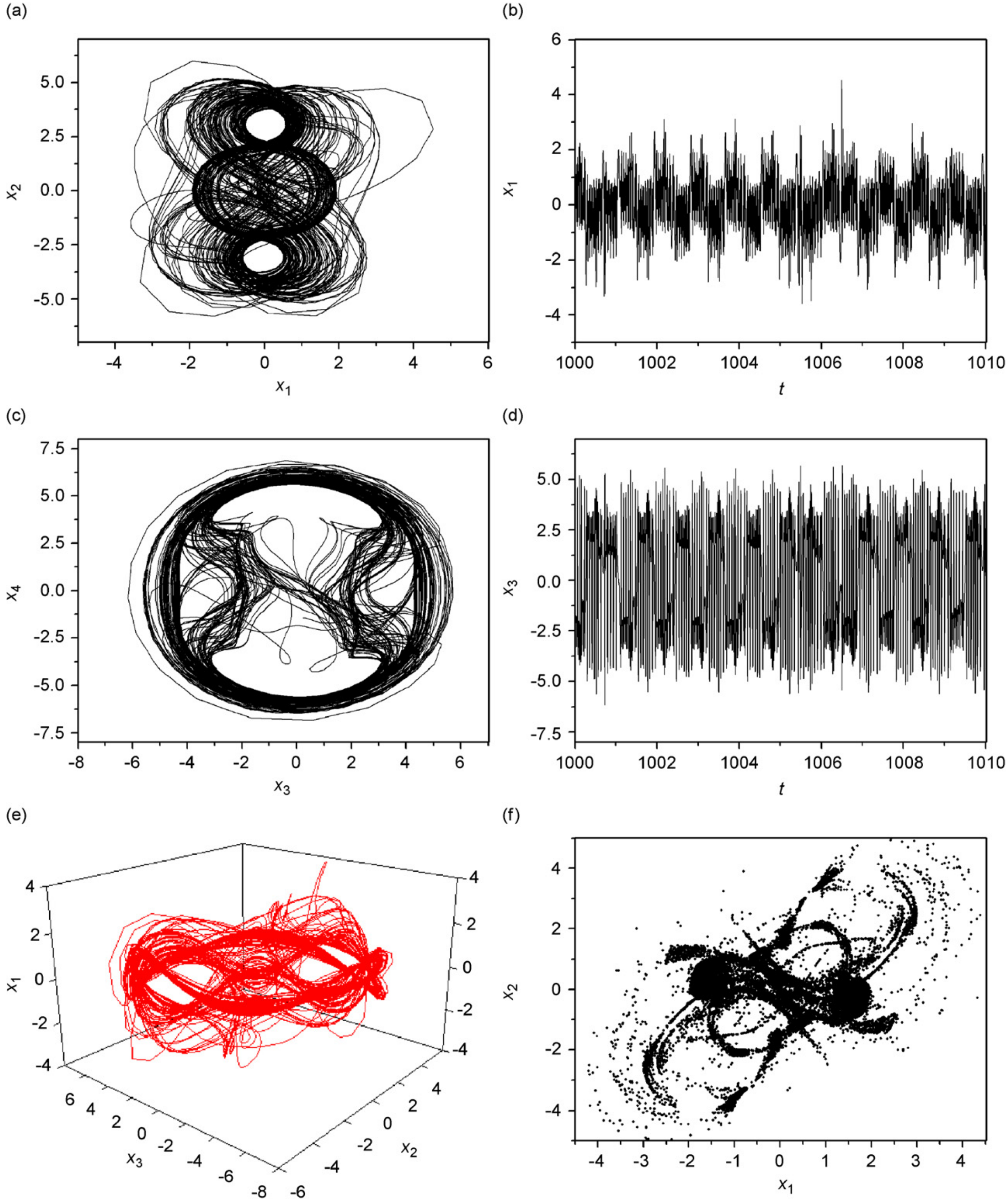

Fig. 5. The chaotic motion of the FGMs rectangular plate exists when $f_2 = 7.038$.

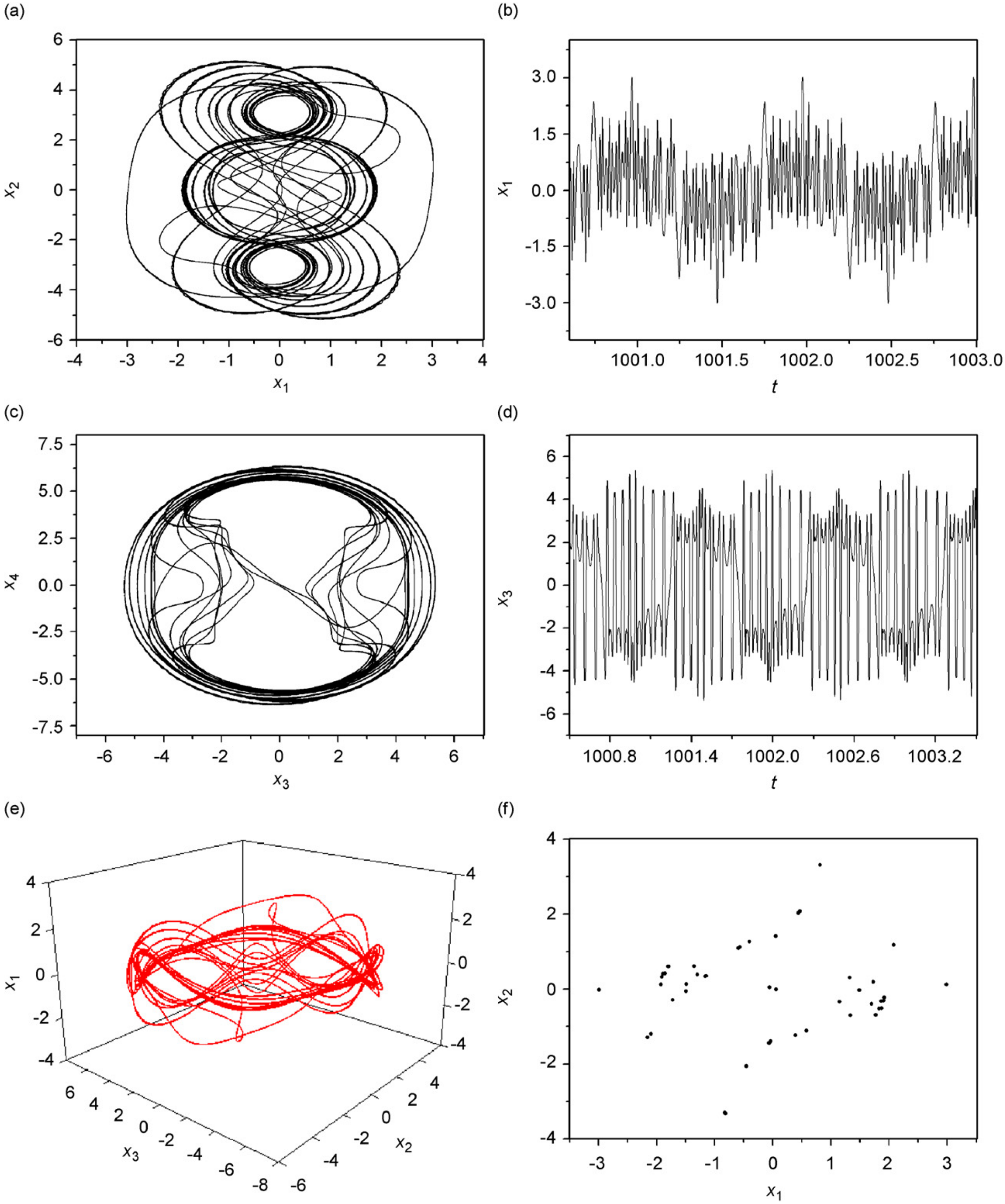

Fig. 6. The quasi-period motion of the FGMs rectangular plate exists when $f_2 = 7.053$.

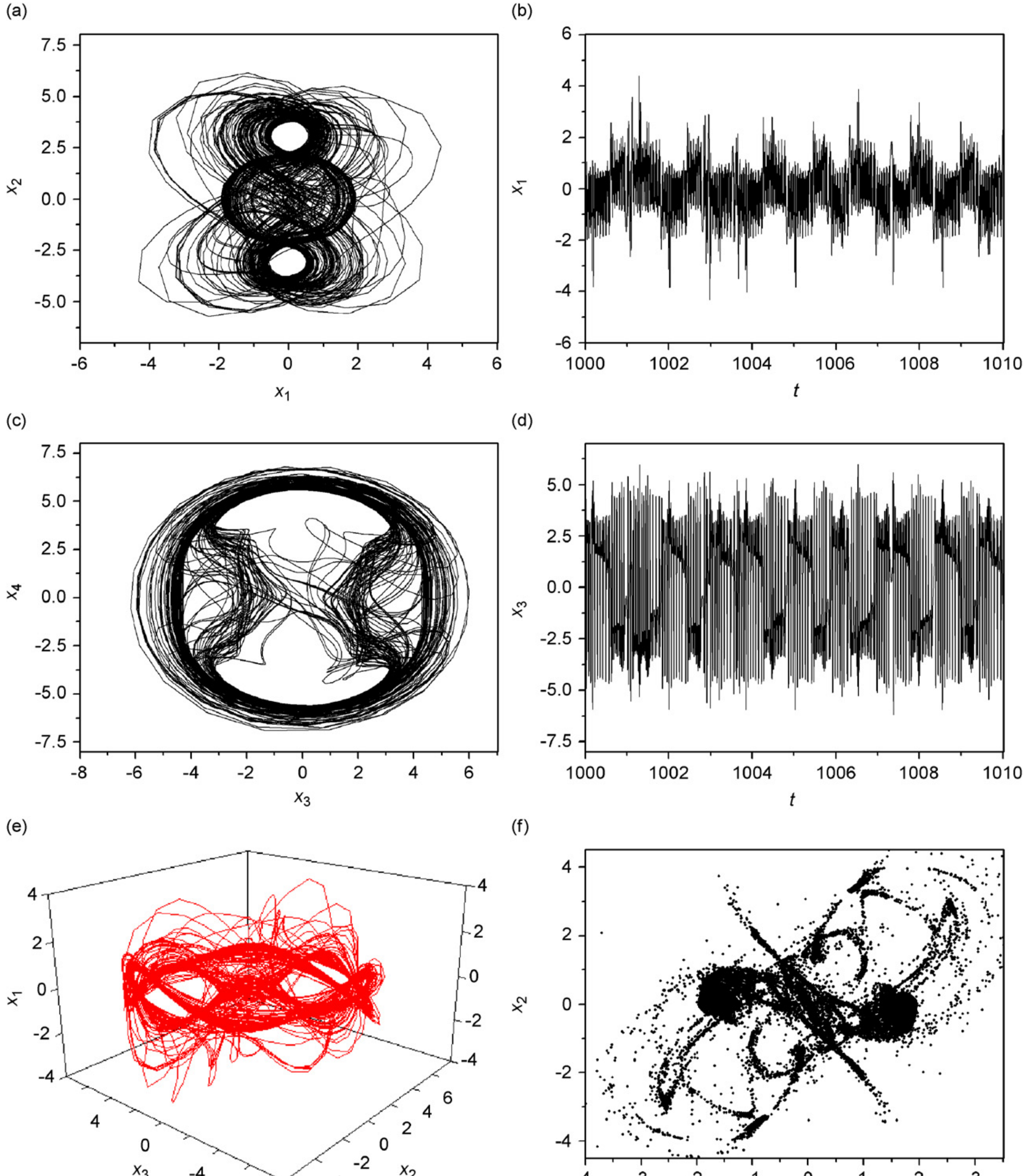

Fig. 7. The chaotic motion of the FGMs rectangular plate exists when $f_2 = 7.098$.

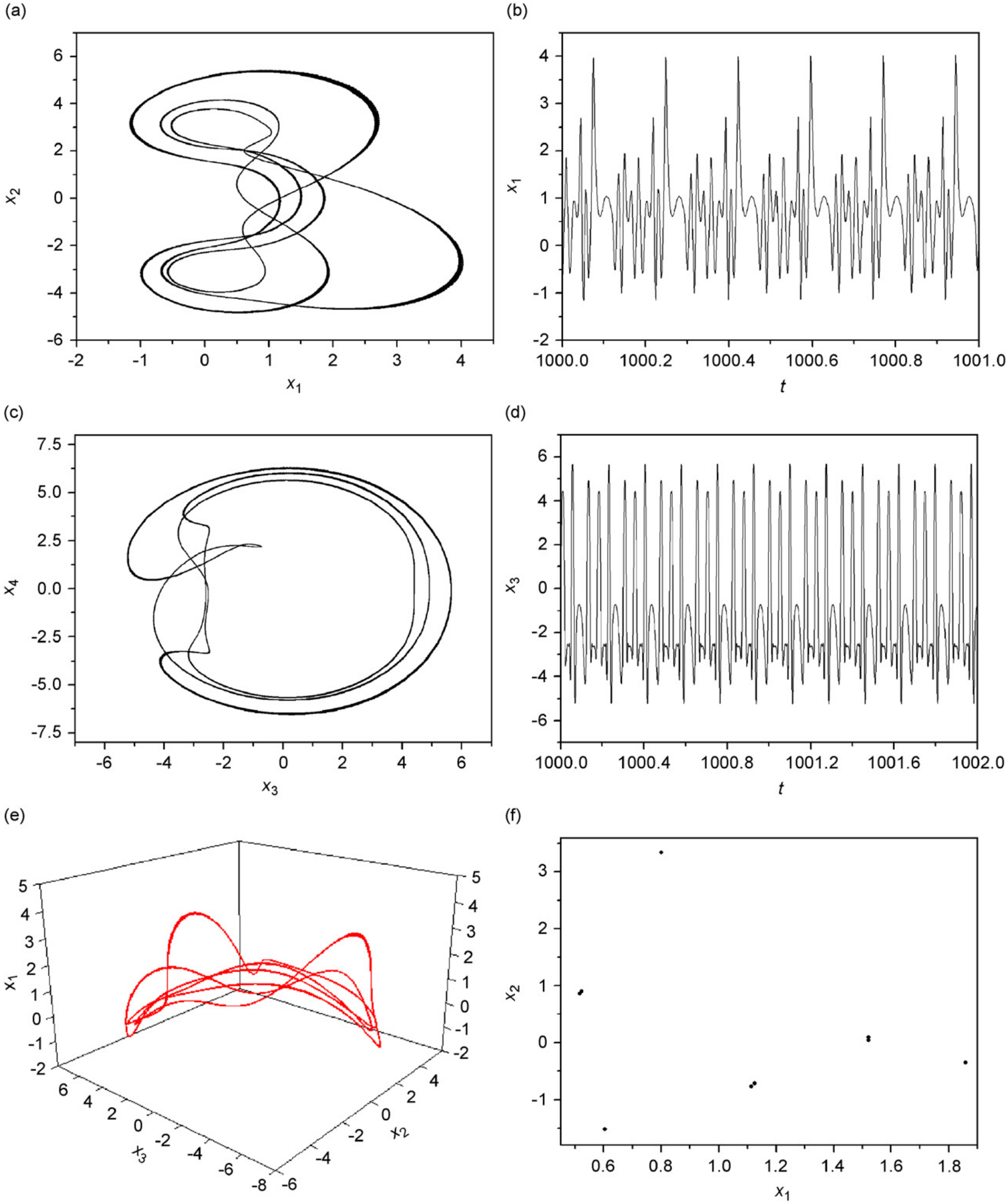

Fig. 8. The period-9 motion of the FGMs rectangular plate exists when $f_2 = 7.288$.

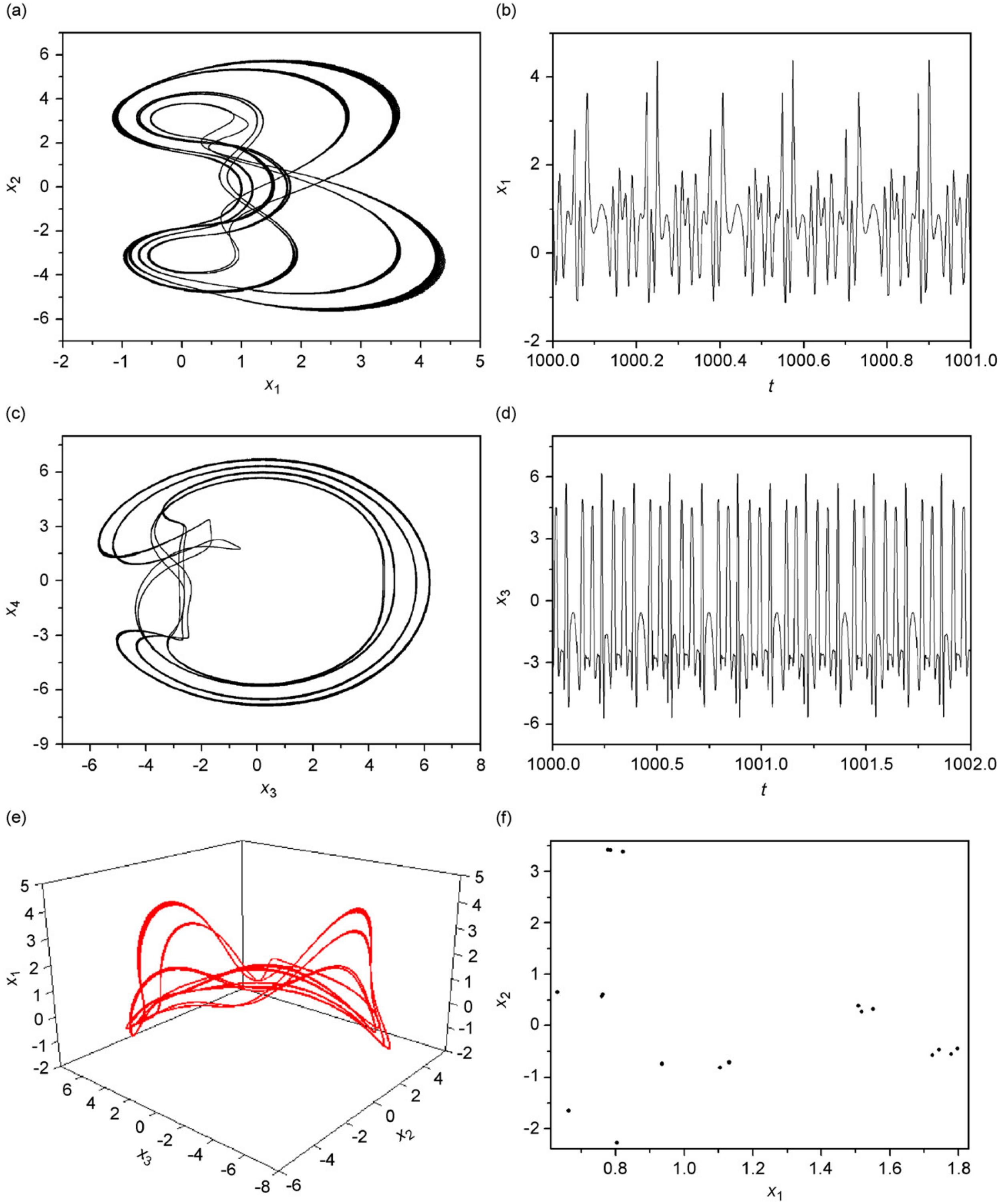

Fig. 9. The multiple period motion of the FGMs rectangular plate exists when $f_2 = 7.389$.

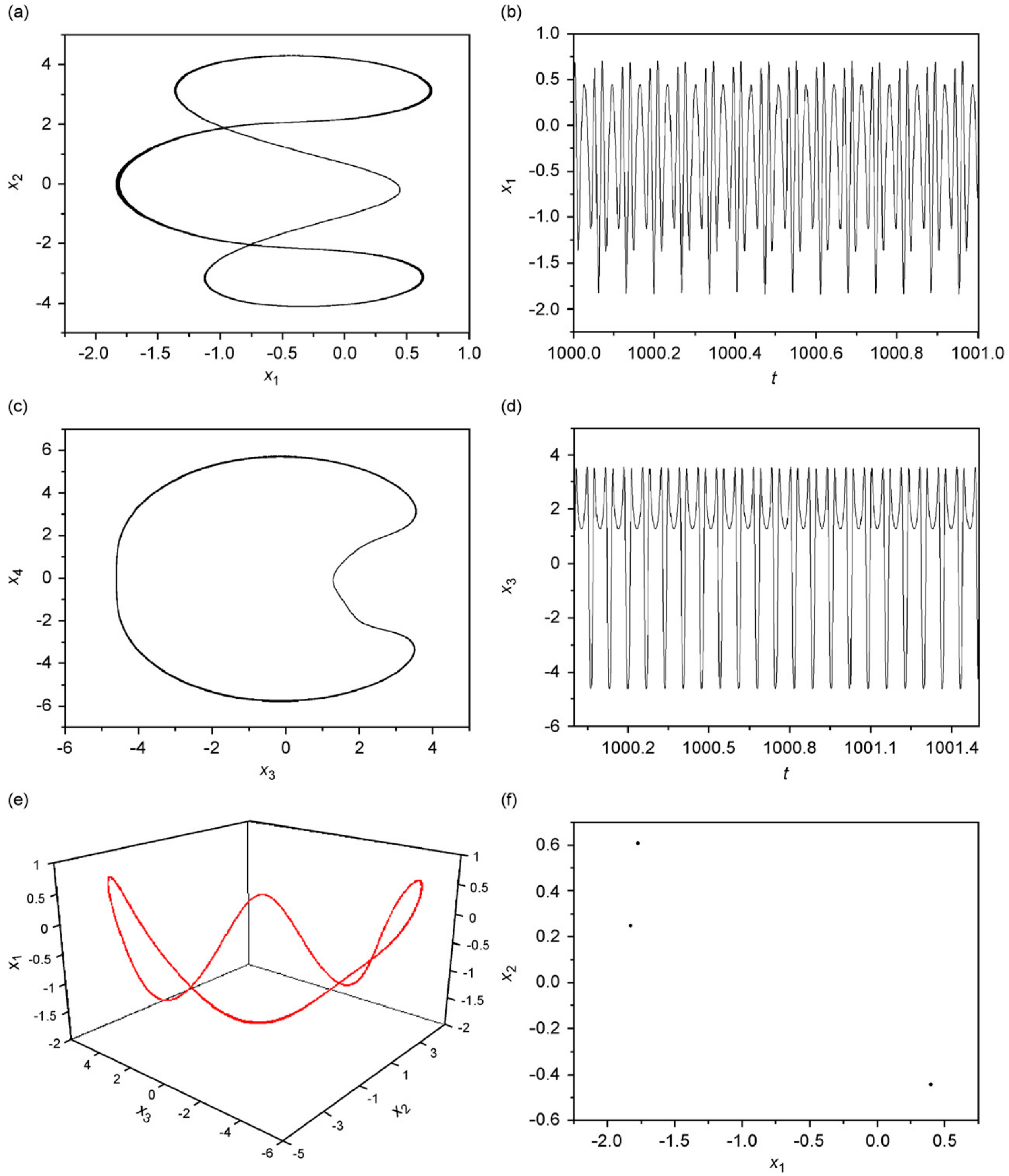

Fig. 10. The period motion of the FGMs rectangular plate exists when $f_2 = 7.403$.

5. Conclusions

The nonlinear oscillations and chaotic dynamics of the FGM rectangular plate under the combined transverse and in-plane excitations are investigated for the first time. The material properties are assumed to be temperature dependent. Based on Reddy's third-order plate theory, the governing equations of motion for the FGM rectangular plate are derived using the Hamilton's principle. Only transverse nonlinear oscillations of the FGM plate are considered. Galerkin's approach is utilized to discretize the governing equation of motion to a two-degree-of-freedom nonlinear system including the quadratic and cubic nonlinear terms. The resonant case considered here is 1:1 internal resonance and principal parametric resonance-1/2 subharmonic resonance. The asymptotic perturbation method based on the Fourier expansion and the temporal rescaling is utilized to obtain a four-dimensional nonlinear averaged equation. Using the fourth-order Runge–Kutta algorithm, the averaged equation is analyzed numerically. Under certain conditions, the periodic, quasi-periodic and chaotic motions of the FGM rectangular plate are found.

The influence of the forcing excitations f_1 and f_2 on the nonlinear dynamic behaviors of the FGM rectangular plate is investigated. It is thought that the forcing excitations f_1 and f_2 can change the form of motions for the FGM rectangular plate. In the situation investigated in this paper, the forcing excitation can be considered to be a controlling force, which can control the responses of the FGM rectangular plate from the chaotic motion to the period n or quasi-period motions.

Because the main interest in this paper is focused on the analytical and numerical researches on the nonlinear oscillations and chaotic dynamics of the FGM rectangular plate under combined transverse and in-plane excitations, we did not find similar work conducted to date. Thus, it is a difficult job for us to give some comparisons with the jobs of other researchers. In this paper, we mainly give the qualitative analysis on nonlinear dynamics of the FGMs rectangular plate rather than the quantitative analysis. Therefore, we did not clearly give the temperature rise ΔT or surface temperature T_1 and T_2. It is thought that the external loads are the fast varying excitations and the temperature rise ΔT or surface temperatures T_1 and T_2 are the slow varying excitations. It is a new problem on how to study the influence of both the fast and slow varying excitations on nonlinear responses of the FGM plate.

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Appendix A

The coefficients a_i $(i = 1, \ldots, 6)$ and b_j $(j = 1, \ldots, 7)$ presented in Eq. (26) are as follows:

$$\omega_1^2 = -\frac{m_{007} + p_0 m_{008}}{m_{001}}, \quad a_1 = -\frac{m_{004}}{m_{001}}, \quad a_2 = -\frac{m_{008} p_1}{m_{001}}, \quad a_3 = -\frac{m_{009}}{m_{001}}, \quad a_4 = -\frac{m_{010}}{m_{001}}, \quad a_5 = -\frac{m_{011}}{m_{001}},$$
$$a_6 = -\frac{m_{012}}{m_{001}}, \quad a_7 = -\frac{m_{014}}{m_{001}}, \quad f_1 = \frac{m_{013}}{m_{001}} F_1, \quad \omega_2^2 = -\frac{n_{007} + P_0 n_{008}}{n_{002}}, \quad b_1 = -\frac{n_{005}}{n_{002}}, \quad b_2 = \frac{n_{008} P_1}{n_{002}},$$
$$b_3 = -\frac{n_{006}}{n_{002}}, \quad b_4 = -\frac{n_{009}}{n_{002}}, \quad b_5 = -\frac{n_{010}}{n_{002}}, \quad b_6 = -\frac{n_{011}}{n_{002}}, \quad b_7 = -\frac{n_{012}}{n_{002}}, \quad f_2 = \frac{n_{013}}{n_{002}} F_2, \tag{A.1}$$

where

$$m_{001} = -3\pi c_1 (I_4 - c_1 I_6)(a\mu_{09} + b\lambda_{09})/4 - \frac{1}{4} I_0 ab, \quad m_{004} = \frac{1}{4}\gamma ab, \quad m_{008} = -\frac{1}{4a} b\pi^2,$$
$$m_{007} = \frac{1}{4}\left[-\frac{b\pi^2}{a}(A_{55} + c_2^2 F_{55} - 2c_2 D_{55}) - \frac{4a\pi^2}{9b}(A_{44} + c_2^2 F_{44} - 2c_2 D_{44}) - \frac{b}{a^3} c_1^2 H_{11}\pi^4 - \frac{81a}{4b^3} c_1^2 H_{22}\pi^4\right.$$
$$\left. - \frac{9\pi^4}{2ab}(c_1^2 H_{21} + 2c_1^2 H_{66}) - \frac{1}{4}\pi b\mu_{09}(A_{55} + c_2^2 F_{55} - 2c_2 D_{55}) + \frac{b}{a^2}\pi^3 \mu_{09} c_1 (F_{11} - c_1 H_{11})\right.$$

$$+\frac{3}{4a}\pi^3\lambda_{09}c_1(F_{12}+2F_{66}-c_1H_{21}-2c_1H_{66})-\frac{3}{4}\pi a\lambda_{09}(A_{44}+c_2^2F_{44}-2c_2D_{44}) + \frac{a}{b^2}\pi^3\lambda_{09}c_1(F_{22}-c_1H_{22})\bigg],$$

$$\begin{aligned}m_{009} = {} & \frac{32b\pi}{4275a^2}c_1E_{11}(-175\lambda_{03}+525\lambda_{07})+\frac{32\pi}{525b}c_1(E_{21}+2E_{66})(-175\lambda_{03})+\frac{32a\pi}{175b^2}c_1E_{22}(-175\mu_{03}) \\ & +\frac{32\pi}{1575a}c_1(E_{21}+2E_{66})(-175\mu_{03})-\frac{4096\pi^2c_1}{3ab}(2E_{66}-E_{12})+\frac{16\pi}{3b}\mu_{09}(B_{21}-c_1E_{21}) \\ & -\frac{8\pi}{3b}\mu_{09}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})-\frac{8\pi}{3b}\mu_{09}(B_{66}-c_1E_{66})+\frac{56b\pi}{171a^2}\mu_{09}(B_{11}-c_1E_{11}) \\ & +\frac{16a}{b^2}\pi\lambda_{09}(B_{22}-c_1E_{22})+\frac{4\pi}{9a}\lambda_{09}(B_{66}-c_1E_{66})+\frac{16}{9a}\pi\lambda_{09}(B_{12}-c_1E_{12})-\frac{8a}{9b^2}\pi\lambda_{09}(B_{22}-c_1E_{22}) \\ & -\frac{8\pi}{9a}\lambda_{09}(B_{66}-c_1E_{66})-\frac{8\pi}{9a}\lambda_{09}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66}),\end{aligned}$$

$$\begin{aligned}m_{010} = {} & \frac{32b\pi}{4275a^2}c_1E_{11}(2178\lambda_{06}+1215\lambda_{08})+\frac{864\pi}{525b}c_1\lambda_{06}(E_{21}+2E_{66})+\frac{96a\pi}{175b^2}c_1E_{22}\mu_{06} \\ & +\frac{7776\pi}{1575a}c_1\mu_{06}(E_{21}+2E_{66})+\frac{4156c_1}{525ab}\pi^2(2E_{66}-E_{12})+\frac{432\pi}{525b}\mu_{10}(B_{21}-c_1E_{21}) \\ & -\frac{1512\pi}{525b}\mu_{10}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})+\frac{21384b}{4275a^2}\pi\mu_{10}(B_{11}-c_1E_{11})+\frac{408\pi}{525b}\mu_{10}(B_{66}-c_1E_{66}) \\ & -\frac{48a\pi}{175b^2}\lambda_{10}(B_{22}-c_1E_{22})+\frac{12852}{1575a}\pi\lambda_{10}(B_{66}-c_1E_{66})-\frac{3888\pi}{1575a}\lambda_{10}(B_{12}-c_1E_{12}) \\ & -\frac{168a}{157b^2}\pi\lambda_{10}(B_{22}-c_1E_{22})+\frac{13608}{1575a}\pi\lambda_{10}(B_{66}-c_1E_{66})+\frac{3672\pi}{1575a}\lambda_{10}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66}),\end{aligned}$$

$$\begin{aligned}m_{011} = {} & \frac{\pi^3}{8b}A_{12}(-\lambda_{01}+\lambda_{02}-2\lambda_{05}+2\lambda_{04})-\frac{b\pi^3}{8a^2}A_{11}(-9\lambda_{01}+9\lambda_{02}-18\lambda_{05}+18\lambda_{04}) \\ & -\frac{b\pi^3}{4a^2}A_{11}(-12\lambda_{05}+3\lambda_{02}-3\lambda_{01}+12\lambda_{04})-\frac{3\pi^3}{8b}A_{66}(-\lambda_{01}-2\lambda_{02}+2\lambda_{05}+\lambda_{04}) \\ & +\frac{\pi^3}{4b}(A_{21}+A_{66})(4\lambda_{05}-2\lambda_{02}-\lambda_{01}+2\lambda_{04})-\frac{3\pi^3}{4b}A_{66}(-4\lambda_{05}+4\lambda_{02}-\lambda_{01}+\lambda_{04}) \\ & +\frac{3\pi^3}{4a}A_{66}(-2\mu_{04}+\mu_{02}+\mu_{01}+2\mu_{05})-\frac{\pi^3}{8a}A_{22}(-\mu_{04}-2\mu_{02}+\mu_{01}+2\mu_{05}) \\ & -\frac{3\pi^3}{8a}A_{12}(-3\mu_{04}-6\mu_{02}+3\mu_{01}6\mu_{05})-\frac{\pi^3a}{4b^2}A_{22}(-\mu_{04}+4\mu_{02}+\mu_{01}-4\mu_{05}) \\ & -\frac{\pi^3}{4a}A_{66}(-4\mu_{04}+\mu_{02}+\mu_{01}-4\mu_{05})+\frac{3\pi^3}{4a}(-2\mu_{04}-2\mu_{02}+\mu_{01}+4\mu_{05})(A_{66}+A_{12}) \\ & -\frac{27\pi^4a}{32b^3}A_{22}+\frac{5\pi^4}{ab}\left(\frac{1}{2}A_{21}+A_{66}\right)-\left(\frac{27\pi^4}{32a^3}+\frac{4b\pi^3}{a^2}\lambda_{08}+\frac{b\pi^4}{a^3}+\frac{9\pi^3}{b}\lambda_{08}+\frac{9\pi^4}{ab}\right)A_{11},\end{aligned}$$

$$\begin{aligned}m_{012} = {} & \frac{\pi^3}{8b}(-9\lambda_{03}+18\lambda_{07})A_{12}+\frac{b\pi^3}{8a^2}(-\lambda_{03}+2\lambda_{07})A_{11}-\frac{b\pi^3}{4a^2}(-\lambda_{03}+2\lambda_{07})A_{11}-\frac{9\pi^3}{8b}\lambda_{03}A_{66} \\ & +\frac{9\pi^3}{4b}(A_{21}+A_{66})\lambda_{03}+\frac{9\pi^3}{4b}\lambda_{03}A_{66}-\frac{3\pi^3}{4a}\mu_{03}A_{66}-\frac{27\pi^3}{8a}\mu_{03}A_{22}-\frac{3\pi^3}{8a}\mu_{03}A_{12}+\frac{27\pi^3a}{4b^2}\mu_{03}A_{22}\end{aligned}$$

$$+\frac{3\pi^3}{4a}\mu_{03}A_{66}+\frac{3\pi^3}{4a}\mu_{03}(A_{66}+A_{12})-\frac{729\pi^4 a}{128b^3}A_{22}-\frac{27\pi^4}{32ab}\left(\frac{1}{2}A_{21}+A_{66}\right)-\frac{9b\pi^4}{128a^3}A_{11}$$
$$-\left(\frac{b\pi^3}{4a^2}\lambda_{07}+\frac{b\pi^4}{16a^3}+\frac{9\pi^3}{4b}\lambda_{07}+\frac{9\pi^4}{16ab}\right)A_{11},$$

$$\begin{aligned}m_{014}=&\frac{32b\pi}{4275a^2}c_1E_{11}(-675\lambda_{02}+1512\lambda_{04}-1080\lambda_{05}+945\lambda_{10})\\&+\frac{32\pi}{525b}c_1(E_{21}+2E_{66})(42\lambda_{04}+105\mu_{01}-1200\mu_{05}-300\lambda_{02})\\&+\frac{32a\pi}{175b^2}c_1E_{22}(7\mu_{04}+35\mu_{01}-40\mu_{05}-200\mu_{02})\\&+\frac{32\pi c_1}{1575a}(E_{21}+2E_{66})(252\mu_{04}+315\mu_{01}-360\mu_{05}-450\mu_{02})\\&-\frac{24912c_1}{525ab}\pi^2(2E_{66}-E_{12})-\frac{16\pi}{525b}(B_{21}-c_1E_{21})(9\mu_{09}+243\mu_{10})\\&+\frac{8\pi}{525b}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})(9\mu_{09}+27\mu_{10})\\&+\frac{8b\pi}{4275a^2}(B_{11}-c_1E_{11})(-729\mu_{09}-2687\mu_{10})+\frac{8\pi}{525b}(B_{66}-c_1E_{66})(729\mu_{09}+27\mu_{10})\\&-\frac{16a\pi}{175b^2}(B_{22}-c_1E_{22})(27\lambda_{10}+9\lambda_{09})-\frac{4\pi}{1575a}(B_{66}-c_1E_{66})(243\lambda_{10}+81\lambda_{09})\\&-\frac{16\pi}{1575a}(B_{12}-c_1E_{12})(27\lambda_{10}+729\lambda_{09})+\frac{8a\pi}{1575b^2}(B_{22}-c_1E_{22})(3\lambda_{10}+9\lambda_{09})\\&-\frac{8\pi}{1575a}(B_{66}-c_1E_{66})(243\lambda_{10}+9\lambda_{09})+\frac{8\pi}{1575a}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})(279\lambda_{09}+243\lambda_{10}),\end{aligned}$$

$$m_{13}=\frac{ab}{4},$$
$$n_{002}=-3\pi c_1(I_4-c_1I_6)(a\mu_{09}+b\lambda_{09})/4-\tfrac{1}{4}I_0ab,\quad n_{005}=\tfrac{1}{4}\gamma ab,$$

$$\begin{aligned}n_{006}=&-\frac{32b\pi}{175a^2}c_1E_{11}(-35\lambda_{01}-7\lambda_{02}+40\lambda_{05}+200\lambda_{04})-\frac{32\pi}{1575b}c_1(E_{21}+2E_{66})(-252\lambda_{02}\\&-315\lambda_{01}+450\lambda_{04}+360\lambda_{05})+\frac{32a\pi}{4725b^2}c_1E_{22}(945\mu_{01}-675\mu_{04}-1080\mu_{05}+1512\mu_{02})\\&+\frac{32\pi}{525a}c_1(E_{12}+2E_{66})(105\mu_{01}+42\mu_{02}-120\mu_{03}-300\mu_{04})-\frac{23636\pi^2}{525ab}c_1(E_{66}-E_{12})\\&-\frac{1269\pi^2}{525ab}c_1(E_{12}-2E_{66})-\frac{16\pi}{1575b}(B_{21}-c_1E_{21})(27\mu_{09}+729\mu_{10})\\&+\frac{8\pi}{1575b}(B_{21}+B_{66}-c_1E_{21}-c_1E_{66})(243\mu_{09}+729\mu_{10})+\frac{8b\pi}{175a^2}(B_{11}-c_1E_{11})(3\mu_{09}+9\mu_{10})\\&-\frac{16b\pi}{175a^2}(B_{11}-c_1E_{11})(27\mu_{09}+9\mu_{10})-\frac{8\pi}{1575b}(B_{66}-c_1E_{66})(243\mu_{09}+9\mu_{10})\\&-\frac{16\pi a}{4725b^2}(B_{22}-c_1E_{22})(243\lambda_{09}+729\lambda_{10})-\frac{8\pi}{525a}(B_{66}-c_1E_{66})(27\lambda_{09}+81\lambda_{10})\\&+\frac{16\pi}{525a}(B_{12}-c_1E_{12})(243\lambda_{09}+9\lambda_{10})+\frac{8a\pi}{4725b^2}(B_{22}-c_1E_{22})(2187\lambda_{09}-729\lambda_{10})\\&+\frac{8\pi}{525a}(B_{66}-c_1E_{66})(27\lambda_{09}+279\lambda_{10})+\frac{8\pi}{525a}(B_{12}+B_{66}-c_1E_{12}-c_1E_{66})(27\lambda_{09}+9\lambda_{10}),\end{aligned}$$

$$n_{007} = -\frac{9b\pi^2}{4a}(A_{55} + c_2^2F_{55} - 2c_2D_{55}) - \frac{a\pi^2}{4b}(A_{44} + c_2^2F_{44} - 2c_2D_{44}) - \frac{81b\pi^4}{4a^3}c_1^2H_{11} - \frac{a\pi^4}{4b^3}c_1^2H_{22}$$
$$- \frac{9\pi^4}{2ab}(c_1^2H_{21} + 2c_1^2H_{66}) - \frac{3}{4}\pi b\mu_{10}(A_{55} + c_2^2F_{55} - 2c_2D_{55}) + \frac{27b}{4a^2}\pi^3\mu_{10}c_1(F_{11} - c_1^2H_{11})$$
$$- \frac{1}{4}\pi a\lambda_{10}(A_{44} + c_2^2F_{44} - 2c_2D_{44}) + \frac{9}{4a}\pi^3\lambda_{10}c_1(F_{12} + 2F_{66} - c_1H_{21} - 2c_1H_{66})$$
$$+ \frac{a}{4b^3}\pi^4\lambda_{10}c_1(F_{22} - c_1H_{22}),$$
$$n_{008} = -\frac{9}{4a}b\pi^2 - \frac{1}{4b}a\pi^2, \quad n_{13} = \frac{ab}{4},$$

$$n_{009} = -\frac{32b\pi}{175a^2}c_1E_{11}(105\lambda_{07} - 3\lambda_{03}) + \frac{7776\pi}{1575b}c_1\lambda_{03}(E_{21} + 2E_{66}) + \frac{69984a\pi}{4725b^2}c_1E_{22}\mu_{03}$$
$$+ \frac{864\pi}{525a}c_1\mu_{03}(E_{12} + 2E_{66}) - \frac{1296\pi^2}{525ab}c_1(E_{66} - E_{12}) - \frac{4284\pi^2}{525ab}c_1(E_{12} - 2E_{66})$$
$$- \frac{3888\pi}{1575b}\mu_{09}(B_{21} - c_1E_{21}) + \frac{3672\pi}{1575b}\mu_{09}(B_{21} + B_{66} - c_1E_{21} - c_1E_{66})$$
$$- \frac{8b\pi}{a^2}\mu_{10}(B_{11} - c_1E_{11}) - \frac{48b\pi}{175a^2}\mu_{09}(B_{11} - c_1E_{11}) - \frac{13608\pi}{1575b}\mu_{09}(B_{66} - c_1E_{66})$$
$$- \frac{34992\pi a}{4725b^2}\lambda_{09}(B_{22} - c_1E_{22}) + \frac{8\pi}{3a}\lambda_{10}(B_{66} - c_1E_{66}) - \frac{432\pi}{525a}\lambda_{09}(B_{12} - c_1E_{12})$$
$$+ \frac{33048a\pi}{4725b^2}\lambda_{09}(B_{22} - c_1E_{22}) + \frac{408\pi}{525a}\lambda_{09}(B_{66} - c_1E_{66}) + \frac{1512\pi}{525a}\lambda_{09}(B_{12} + B_{66} - c_1E_{12} - c_1E_{66}),$$

$$n_{010} = -\frac{32b\pi}{175a^2}c_1E_{11}(175\lambda_{06} - 525\lambda_{08}) - \frac{224\pi}{63b}c_1\lambda_{06}(E_{21} + 2E_{66}) - \frac{32a\pi}{27b^2}c_1E_{22}\mu_{06}$$
$$- \frac{32\pi}{3a}\mu_{06}c_1(E_{12} + 2E_{66}) - \frac{8\pi^2}{3ab}c_1(2E_{66} - E_{12}) - \frac{42\pi^2}{ab}c_1(E_{12} - 2E_{66})$$
$$+ \frac{16\pi}{9b}\mu_{10}(B_{21} - c_1E_{21}) - \frac{8\pi}{9b}\mu_{10}(B_{21} + B_{66} - c_1E_{21} - c_1E_{66}) - \frac{168b\pi}{175a^2}\mu_{09}(B_{11} - c_1E_{11})$$
$$+ \frac{16b\pi}{a^2}\mu_{09}(B_{11} - c_1E_{11}) - (B_{66} - c_1E_{66})\frac{14008\pi}{1575b}\mu_{09} + \frac{9\pi^4}{23ab}(A_{21} + 2A_{66}) + \frac{16\pi a}{27b^2}\lambda_{10}(B_{22} - c_1E_{22})$$
$$+ \frac{120\pi}{21a}\lambda_{09}(B_{66} - c_1E_{66}) + \frac{16\pi}{3a}\lambda_{10}(B_{12} - c_1E_{12}) - \frac{8a\pi}{27b^2}\lambda_{10}(B_{22} - c_1E_{22}) - \frac{8\pi}{3a}\lambda_{09}(B_{66} - c_1E_{66})$$
$$- \frac{8\pi}{3a}\lambda_{10}(B_{12} + B_{66} - c_1E_{12} - c_1E_{66}),$$

$$n_{011} = -\frac{3}{8b}\pi^3A_{21}(-6\lambda_{04} + 3\lambda_{01} - 3\lambda_{02} + 6\lambda_{05}) - \frac{b}{8a^2}\pi^3A_{11}(2\lambda_{05} - 2\lambda_{04} + \lambda_{01} - \lambda_{02})$$
$$+ \frac{3}{4b}\pi^3A_{66}(2\lambda_{05} - \lambda_{04} - \lambda_{01} + 2\lambda_{02}) + \frac{3}{4b}\pi^3(4\lambda_{05} - 2\lambda_{04} + \lambda_{01} - 2\lambda_{02})(A_{66} + A_{21})$$
$$+ \frac{b}{4a^2}\pi^3A_{11}(4\lambda_{05} - 4\lambda_{04} - \lambda_{01} + \lambda_{02}) + \frac{1}{4b}\pi^3A_{66}(4\lambda_{05} - \lambda_{04} - \lambda_{01} + 4\lambda_{02})$$
$$+ \frac{3}{4a}\pi^3A_{66}(2\mu_{01} + 2\mu_{04} - \mu_{02} - 2\mu_{05}) - \frac{1}{8a}\pi^3A_{12}(\mu_{01} - \mu_{04} - 2\mu_{02} + 2\mu_{05})$$
$$+ \frac{3}{4a}\pi^3A_{66}(2\mu_{01} + 2\mu_{04} - \mu_{02} - 2\mu_{05}) - \frac{1}{4a}\pi^3(\mu_{01} + 2\mu_{04} - 2\mu_{02} - 4\mu_{05})(A_{66} + A_{12})$$

$$
\begin{aligned}
&-\frac{27a\pi^4}{32b^3}A_{22}-\frac{\pi^4}{16ab}\left(\frac{1}{2}A_{12}+A_{66}\right)-\frac{81\pi^4}{16ab}\left(\frac{1}{2}A_{12}+A_{66}\right)-\frac{729b\pi^4}{128a^3}A_{11}\\
&-\left(\frac{9b}{4a^2}\pi^3\lambda_{07}+\frac{9b}{16a^3}\pi^4+\frac{1}{4ab}\pi^3\lambda_{07}+\frac{1}{16ab}\pi^4\right)A_{11}\\
&-\frac{a\pi^3}{8b^2}A_{22}(9\mu_{01}-18\mu_{02}+18\mu_{05}-9\mu_{04})+\frac{a\pi^3}{4b^2}A_{22}(3\mu_{01}-12\mu_{02}+12\mu_{05}-3\mu_{04}),
\end{aligned}
$$

$$
\begin{aligned}
n_{012}=&-\frac{3}{8b}\pi^3(6\lambda_{06}-2\lambda_{08})A_{21}-\frac{b}{8a^2}\pi^3(27\lambda_{06}-54\lambda_{08})A_{11}-\frac{3}{4b}\pi^3\lambda_{06}A_{66}+\frac{3}{4b}\pi^3\lambda_{06}(A_{66}+A_{21})\\
&-\frac{b}{4a^2}\pi^3(54\lambda_{08}-27\lambda_{06})A_{11}+\frac{3}{4b}\pi^3\lambda_{06}A_{66}-\frac{9}{4a}\pi^3\mu_{06}A_{66}+\frac{a}{8b^2}\pi^3\mu_{06}A_{22}-\frac{9}{8a}\pi^3\mu_{06}A_{12}\\
&+\frac{9}{4a}\pi^3\mu_{06}(A_{66}+A_{12})-\frac{9a\pi^4}{128b^3}A_{22}-\frac{27}{64ab}\pi^4\left(\frac{1}{2}A_{12}+A_{66}\right)-\frac{27}{64ab}\pi^4\left(\frac{1}{2}A_{12}+A_{66}\right)\\
&-\frac{27b\pi^4}{32a^3}A_{11}-\left(\frac{9b}{a^2}\pi^3\lambda_{08}+\frac{9b}{4a^3}\pi^4+\frac{1}{ab}\pi^3\lambda_{08}+\frac{1}{4ab}\pi^4\right)A_{11},
\end{aligned}
\tag{A.2}
$$

where

$$
\begin{aligned}
&\mu_i=\frac{g_{i2}l_{i1}-g_{i1}l_{i2}}{g_{i2}k_{i1}-g_{i1}k_{i2}},\quad \lambda_i=\frac{k_{i2}l_{i1}-k_{i1}l_{i2}}{k_{i2}g_{i1}-k_{i1}g_{i2}}\quad (i=1,\ldots,10),\\
&g_{i1}=\sum_{j=1}^{4}(\xi_{1j})^2,\quad g_{i2}=\sum_{j=1}^{4}\xi_{1j}\zeta_{1j},\quad k_{i2}=\sum_{j=1}^{4}(\zeta_{1j})^2,\quad k_{i1}=\sum_{j=1}^{4}\xi_{1j}\zeta_{1j},\\
&l_{i1}=\sum_{j=1}^{4}\xi_{1j}\eta_{1j},\quad l_{i2}=\sum_{j=1}^{4}\zeta_{1j}\eta_{1j}\quad (i=1,\ldots,10),
\end{aligned}
\tag{A.3}
$$

$$
\begin{aligned}
&\xi_{11}=4\pi^2\left(\frac{1}{a^2}q_{11}+\frac{1}{b^2}q_{12}\right),\quad \xi_{12}=\frac{4\pi^2}{ab}q_{23},\quad \xi_{13}=4\pi^2\left(\frac{1}{a^2}q_{31}+\frac{1}{b^2}q_{32}\right),\quad \xi_{14}=4\frac{\pi^2}{ab}q_{43},\\
&\xi_{21}=4\pi^2\left(\frac{1}{a^2}q_{11}+\frac{4}{b^2}q_{12}\right),\quad \xi_{22}=\frac{8\pi^2}{ab}q_{23},\quad \xi_{23}=4\pi^2\left(\frac{1}{a^2}q_{31}+\frac{4}{b^2}q_{32}\right),\quad \xi_{24}=\frac{8\pi^2}{ab}q_{43},\\
&\xi_{31}=4\pi^2\left(\frac{1}{a^2}q_{11}+\frac{9}{b^2}q_{12}\right),\quad \xi_{32}=\frac{12\pi^2}{ab}q_{23},\quad \xi_{33}=4\pi^2\left(\frac{1}{a^2}q_{31}+9q_{32}\right),\quad \xi_{34}=\frac{12\pi^2}{ab}q_{43},\\
&\xi_{41}=4\pi^2\left(\frac{4}{a^2}q_{11}+\frac{1}{b^2}q_{12}\right),\quad \xi_{42}=\frac{8\pi^2}{ab}q_{23},\quad \xi_{43}=4\pi^2\left(\frac{4}{a^2}q_{31}+\frac{1}{b^2}q_{32}\right),\quad \xi_{44}=\frac{8\pi^2}{ab}q_{43},\\
&\xi_{51}=16\pi^2\left(\frac{1}{a^2}q_{11}+\frac{1}{b^2}q_{12}\right),\quad \xi_{52}=\frac{16\pi^2}{ab}q_{23},\quad \xi_{53}=16\pi^2\left(\frac{1}{a^2}q_{31}+\frac{1}{b^2}q_{32}\right),\quad \xi_{54}=\frac{16\pi^2}{ab}q_{43},
\end{aligned}
$$

$$
\begin{aligned}
&\xi_{61}=4\pi^2\left(\frac{9}{a^2}q_{11}+\frac{1}{b^2}q_{12}\right),\quad \xi_{62}=\frac{12\pi^2}{ab}q_{23},\quad \xi_{63}=4\pi^2\left(\frac{9}{a^2}q_{31}+\frac{1}{b^2}q_{32}\right),\quad \xi_{64}=\frac{12\pi^2}{ab}q_{43},\\
&\xi_{71}=\frac{4\pi^2}{a^2}q_{11},\quad \xi_{72}=0,\quad \xi_{73}=\frac{4\pi^2}{a^2}q_{31},\quad \xi_{74}=0,\quad \xi_{81}=\frac{36\pi^2}{a^2}q_{11},\quad \xi_{82}=0,\\
&\xi_{83}=\frac{36\pi^2}{a^2}q_{31},\quad \xi_{84}=0,\quad \xi_{91}=\pi^2\left(\frac{1}{a^2}q_{14}+\frac{9}{b^2}q_{15}\right),\quad \xi_{92}=\frac{3\pi^2}{ab}q_{26},\\
&\xi_{93}=\pi^2\left(\frac{1}{a^2}q_{34}+\frac{9}{b^2}q_{35}\right)+q_{37},\quad \xi_{94}=\frac{3\pi^2}{ab}q_{46},\quad \xi_{101}=\pi^2\left(\frac{9}{a^2}q_{14}+\frac{1}{b^2}q_{15}\right),\\
&\xi_{102}=\frac{3\pi^2}{ab}q_{26},\quad \xi_{103}=\pi^2\left(\frac{9}{a^2}q_{34}+\frac{1}{b^2}q_{35}\right)+q_{37},\quad \xi_{104}=\frac{3\pi^2}{ab}q_{46},\quad \zeta_{11}=\frac{4\pi^2}{ab}q_{13},
\end{aligned}
$$

$$\zeta_{12}=4\pi^2\left(\frac{1}{a^2}q_{21}+\frac{1}{b^2}q_{22}\right),\quad \zeta_{13}=\frac{4\pi^2}{ab}q_{33},\quad \zeta_{14}=4\pi^2\left(\frac{1}{a^2}q_{41}+\frac{1}{b^2}q_{42}\right),\quad \zeta_{21}=\frac{8\pi^2}{ab}q_{13},$$
$$\zeta_{22}=4\pi^2\left(\frac{1}{a^2}q_{21}+\frac{4}{b^2}q_{22}\right),\quad \zeta_{23}=\frac{8\pi^2}{ab}q_{33},\quad \zeta_{24}=4\pi^2\left(\frac{1}{a^2}q_{41}+\frac{4}{b^2}q_{42}\right),\quad \zeta_{31}=\frac{12\pi^2}{ab}q_{13},$$
$$\zeta_{32}=4\pi^2\left(\frac{1}{a^2}q_{21}+\frac{9}{b^2}q_{22}\right),\quad \zeta_{33}=\frac{12\pi^2}{ab}q_{33},\quad \zeta_{34}=4\pi^2\left(\frac{1}{a^2}q_{41}+\frac{9}{b^2}q_{42}\right),\quad \zeta_{41}=\frac{8\pi^2}{ab}q_{13},$$
$$\zeta_{42}=4\pi^2\left(\frac{4}{a^2}q_{21}+\frac{1}{b^2}q_{22}\right),\quad \zeta_{43}=\frac{8\pi^2}{ab}q_{33},\quad \zeta_{44}=4\pi^2\left(\frac{4}{a^2}q_{41}+\frac{1}{b^2}q_{42}\right),\quad \zeta_{51}=\frac{12\pi^2}{ab}q_{13},$$
$$\zeta_{52}=\pi^2\left(\frac{16}{a^2}q_{21}+\frac{16}{b^2}q_{22}\right),\quad \zeta_{53}=\frac{16\pi^2}{ab}q_{33},\quad \zeta_{54}=16\pi^2\left(\frac{1}{a^2}q_{41}+\frac{1}{b^2}q_{42}\right),\quad \zeta_{61}=\frac{12\pi^2}{ab}q_{13},$$

$$\zeta_{62}=4\pi^2\left(\frac{9}{a^2}q_{21}+\frac{1}{b^2}q_{22}\right),\quad \zeta_{63}=\frac{12\pi^2}{ab}q_{33},\quad \zeta_{64}=4\pi^2\left(\frac{9}{a^2}q_{41}+\frac{1}{b^2}q_{42}\right),\quad \zeta_{71}=0,$$
$$\zeta_{72}=\frac{4\pi^2}{b^2}q_{22},\quad \zeta_{73}=0,\quad \zeta_{74}=\frac{4\pi^2}{b^2}q_{42},\quad \zeta_{81}=0,\quad \zeta_{82}=\frac{36\pi^2}{b^2}q_{22},\quad \zeta_{83}=0,$$
$$\zeta_{84}=\frac{36\pi^2}{b^2}q_{42},\quad \zeta_{91}=\frac{3\pi^2}{ab}q_{16},\quad \zeta_{92}=\pi^2\left(\frac{1}{a^2}q_{24}+\frac{9}{b^2}q_{25}\right),\quad \zeta_{93}=\frac{3\pi^2}{ab}q_{36},$$
$$\zeta_{94}=\pi^2\left(\frac{1}{a^2}q_{44}+\frac{9}{b^2}q_{45}\right)+q_{47},\quad \zeta_{101}=\frac{3\pi^2}{ab}q_{16},\quad \zeta_{102}=4\pi^2\left(\frac{9}{a^2}q_{24}+\frac{1}{b^2}q_{25}\right),$$
$$\zeta_{103}=\frac{3\pi^2}{ab}q_{36},\quad \zeta_{104}=\pi^2\left(\frac{9}{a^2}q_{44}+\frac{1}{b^2}q_{45}\right)-q_{47},\quad \eta_{11}=\frac{\pi^3}{2a}\left(\frac{-3}{a^2}s_{11}+\frac{14}{b^2}s_{12}-\frac{3}{b^2}s_{13}\right),$$

$$\eta_{12}=\frac{\pi^3}{2b}\left(\frac{14}{a^2}s_{21}-\frac{3}{b^2}s_{22}+\frac{3}{a^2}s_{23}\right),\quad \eta_{13}=\frac{\pi^3}{2a}\left(\frac{-3}{a^2}s_{31}+\frac{14}{b^2}s_{32}+\frac{-3}{b^2}s_{33}\right),$$
$$\eta_{14}=\frac{\pi^3}{2b}\left(\frac{14}{a^2}s_{41}+\frac{-3}{b^2}s_{42}+\frac{3}{a^2}s_{43}\right),\quad \eta_{21}=-\frac{\pi^3}{2a}\left(\frac{3}{a^2}s_{11}+\frac{13}{b^2}s_{12}+\frac{6}{b^2}s_{13}\right),$$
$$\eta_{22}=\frac{\pi^3}{b}\left(\frac{-7}{a^2}s_{21}-\frac{3}{b^2}s_{22}+\frac{3}{a^2}s_{23}\right),\quad \eta_{23}=\frac{\pi^3}{2a}\left(\frac{3}{a^2}s_{31}+\frac{13}{b^2}s_{32}+\frac{6}{b^2}s_{33}\right),$$
$$\eta_{24}=\frac{\pi^3}{b}\left(\frac{-2}{a^2}s_{41}-\frac{3}{b^2}s_{42}+\frac{3}{a^2}s_{43}\right),\quad \eta_{31}=\frac{\pi^3}{4a}\left(\frac{1}{a^2}s_{11}+\frac{9}{b^2}s_{12}+\frac{3}{b^2}s_{13}\right),$$
$$\eta_{32}=-\frac{\pi^3}{4b}\left(\frac{3}{a^2}s_{21}+\frac{27}{b^2}s_{22}+\frac{9}{a^2}s_{23}\right),\quad \eta_{33}=\frac{\pi^3}{a}\left(\frac{1}{a^2}s_{31}+\frac{9}{b^2}s_{32}+\frac{3}{b^2}s_{33}\right),$$

$$\eta_{34}=\frac{\pi^3}{4b}\left(\frac{3}{a^2}s_{41}+\frac{27}{b^2}s_{42}+\frac{9}{a^2}s_{43}\right),\quad \eta_{41}=-\frac{\pi^3}{a}\left(\frac{3}{a^2}s_{11}+\frac{7}{b^2}s_{12}+\frac{3}{2b^2}s_{13}\right),$$
$$\eta_{42}=\frac{\pi^3}{4b}\left(-\frac{13}{a^2}s_{21}+\frac{6}{b^2}s_{22}+\frac{3}{a^2}s_{23}\right),\quad \eta_{43}=-\frac{\pi^3}{a}\left(\frac{3}{a^2}s_{31}+\frac{7}{b^2}s_{32}+\frac{3}{2b^2}s_{33}\right),$$
$$\eta_{44}=\frac{\pi^3}{2b}\left(-\frac{13}{a^2}s_{41}+\frac{3}{b^2}s_{42}+\frac{3}{a^2}s_{43}\right),\quad \eta_{51}=\frac{\pi^3}{a}\left(\frac{3}{a^2}s_{11}+\frac{7}{b^2}s_{12}+\frac{3}{b^2}s_{13}\right),$$
$$\eta_{52}=\frac{\pi^3}{b}\left(\frac{7}{a^2}s_{21}+\frac{3}{b^2}s_{22}+\frac{3}{a^2}s_{23}\right),\quad \eta_{53}=\frac{\pi^3}{a}\left(\frac{3}{a^2}s_{31}+\frac{7}{b^2}s_{32}+\frac{3}{b^2}s_{33}\right),$$
$$\eta_{54}=\frac{\pi^3}{b}\left(\frac{7}{a^2}s_{41}+\frac{3}{b^2}s_{42}+\frac{3}{a^2}s_{43}\right),\quad \eta_{61}=\frac{\pi^3}{4a}\left(\frac{27}{a^2}s_{11}+\frac{3}{b^2}s_{12}+\frac{9}{b^2}s_{13}\right),$$

$$\eta_{62} = \frac{\pi^3}{4b}\left(\frac{9}{a^2}s_{21} + \frac{1}{b^2}s_{22} + \frac{3}{a^2}s_{23}\right), \quad \eta_{63} = \frac{\pi^3}{4a}\left(\frac{27}{a^2}s_{31} + \frac{3}{b^2}s_{32} + \frac{9}{b^2}s_{33}\right),$$
$$\eta_{64} = \frac{\pi^3}{4b}\left(\frac{9}{a^2}s_{41} + \frac{1}{b^2}s_{42} + \frac{3}{a^2}s_{43}\right), \quad \eta_{71} = \frac{\pi^3}{4a}\left(-\frac{1}{a^2}s_{11} - \frac{9}{b^2}s_{12} + \frac{3}{b^2}s_{13}\right),$$
$$\eta_{72} = \frac{\pi^3}{4b}\left(\frac{-3}{a^2}s_{21} - \frac{27}{b^2}s_{22} + \frac{9}{a^2}s_{23}\right), \quad \eta_{73} = \frac{\pi^3}{4a}\left(-\frac{1}{a^2}s_{31} - \frac{9}{b^2}s_{32} + \frac{3}{b^2}s_{33}\right),$$
$$\eta_{74} = \frac{\pi^3}{4b}\left(-\frac{3}{a^2}s_{41} - \frac{27}{b^2}s_{42} + \frac{9}{a^2}s_{43}\right), \quad \eta_{81} = \frac{\pi^3}{4a}\left(-\frac{27}{a^2}s_{11} - \frac{3}{b^2}s_{12} + \frac{9}{b^2}s_{13}\right),$$
$$\eta_{82} = \frac{\pi^3}{4b}\left(-\frac{9}{a^2}s_{21} - \frac{1}{b^2}s_{22} + \frac{3}{a^2}s_{23}\right), \quad \eta_{83} = \frac{\pi^3}{4a}\left(-\frac{27}{a^2}s_{31} - \frac{3}{b^2}s_{32} + \frac{9}{b^2}s_{33}\right),$$

$$\eta_{84} = \frac{\pi^3}{4b}\left(-\frac{9}{a^2}s_{41} - \frac{1}{b^2}s_{42} + \frac{3}{a^2}s_{43}\right), \quad \eta_{91} = -\frac{\pi^3}{a}\left(\frac{1}{a^2}s_{14} + \frac{9}{b^2}s_{15}\right),$$
$$\eta_{92} = -\frac{\pi^3}{b}\left(\frac{27}{b^2}s_{24} + \frac{3}{a^2}s_{25}\right), \quad \eta_{93} = -\frac{\pi}{a}\left(\frac{\pi^2}{a^2}s_{34} + \frac{9\pi^2}{b^2}s_{35} - s_{36}\right),$$
$$\eta_{94} = -\frac{\pi}{b}\left(\frac{\pi^2}{b^2}s_{44} + \frac{9\pi^2}{a^2}s_{45} - s_{46}\right), \quad \eta_{101} = \frac{\pi^3}{a}\left(\frac{27}{a^2}s_{14} + \frac{3}{b^2}s_{15}\right),$$
$$\eta_{102} = \frac{\pi^3}{b}\left(\frac{1}{b^2}s_{24} + \frac{9}{a^2}s_{25}\right), \quad \eta_{103} = \frac{\pi}{a}\left(\frac{27\pi^2}{a^2}s_{34} + \frac{3\pi^2}{b^2}s_{35} + 3s_{36}\right),$$
$$\eta_{104} = \frac{\pi}{b}\left(\frac{\pi^2}{b^2}s_{44} + \frac{9\pi^2}{a^2}s_{45} + s_{46}\right), \tag{A.4}$$

and

$$q_{11} = s_{11} = A_{11}, \quad q_{12} = q_{21} = s_{12} = s_{21} = A_{66}, \quad q_{13} = q_{23} = s_{13} = s_{23} = A_{12} + A_{66},$$
$$q_{14} = q_{25} = q_{31} = q_{42} = s_{31} = s_{42} = B_{11} - c_1E_{11}, \quad s_{15} = s_{25} = -c_1E_{12} - 2c_1E_{66},$$
$$q_{15} = q_{24} = q_{32} = q_{42} = s_{32} = s_{41} = B_{66} - c_1E_{66}, \quad q_{22} = s_{22} = A_{22}, \quad s_{14} = s_{24} = -c_1E_{11},$$
$$q_{16} = q_{26} = q_{32} = q_{33} = q_{43} = s_{33} = s_{43} = B_{12} - c_1E_{12} + B_{66} - c_1E_{66},$$
$$q_{34} = q_{45} = D_{11} - 2c_1F_{11} + c_1^2H_{11}, \quad q_{35} = q_{44} = D_{66} - 2c_1F_{66} + c_1^2H_{66},$$
$$q_{36} = q_{47} = s_{36} = s_{46} = -(A_{55} - 2c_1D_{55} + c_2^2F_{55}), \quad s_{34} = s_{44} = -c_1F_{11} + c_1^2H_{11},$$
$$s_{35} = s_{45} = -c_1F_{12} - 2c_1F_{66} + c_1^2H_{12} + 2c_1^2H_{66},$$
$$q_{37} = q_{46} = D_{12} - 2c_1F_{12} - 2c_1F_{66} + D_{66} + c_1^2H_{12} + c_1^2H_{66}. \tag{A.5}$$

Appendix B

The coefficients given in Eq. (43) are as follows:

$$\mu_1 = -\frac{1}{2}a_1, \quad \alpha_1 = -\frac{1}{2\Omega}a_2, \quad \alpha_2 = -\frac{4}{3\Omega}a_3, \quad \alpha_3 = -\frac{4}{3\Omega}a_4, \quad \alpha_4 = -\frac{40}{3\Omega^3}a_3^2 - \frac{3}{\Omega}a_6,$$
$$\alpha_5 = -\frac{16}{\Omega^3}a_3a_4 + \frac{16}{3\Omega^3}a_4b_3 + \frac{2}{\Omega}a_5 - \frac{8}{\Omega^3}a_7b_5 - \frac{8}{3\Omega^3}a_7^2, \quad \alpha_7 = -\frac{20}{3\Omega^3}a_7a_4 - \frac{40}{3\Omega^3}a_4b_5,$$
$$\alpha_6 = \frac{16}{\Omega^3}a_4b_4 + \frac{8}{3\Omega^3}a_3a_7 - \frac{16}{3\Omega^3}a_7b_3, \quad \alpha_9 = -\frac{20}{3\Omega^3}a_3a_4 + \frac{8}{3\Omega^3}a_4b_4 - \frac{4}{\Omega^3}a_7b_3,$$
$$\alpha_8 = \frac{8}{3\Omega^3}a_3a_4 + \frac{4}{3\Omega^3}a_7b_5 + \frac{8}{\Omega^3}a_4b_3 + \frac{1}{\Omega}a_5, \quad \alpha_{10} = \frac{4}{\Omega}a_7^2, \quad \alpha_{11} = -\frac{2}{3\Omega}a_7, \quad \mu_2 = -\frac{1}{2}b_1,$$
$$\beta_1 = \frac{1}{2\Omega}b_2, \quad \beta_2 = -\frac{2}{3\Omega^3}b_3, \quad \beta_3 = -\frac{4}{3\Omega^3}b_5, \quad \beta_4 = \frac{2}{3\Omega^3}b_3, \quad \beta_5 = \frac{4}{3\Omega^3}b_4,$$

$$\beta_6 = -\frac{8}{\Omega^3}a_3b_3 + \frac{16}{3\Omega^3}b_3^2 - \frac{16}{\Omega^3}b_4b_5 - \frac{2}{\Omega}b_6 + \frac{32}{3\Omega^3}a_7b_4, \quad \beta_7 = -\frac{20}{\Omega^3}a_4b_3 - \frac{40}{3\Omega^3}b_5^2 + \frac{3}{\Omega}b_7,$$
$$\beta_8 = -\frac{20}{3\Omega^3}b_4b_3 - \frac{40}{3\Omega^3}a_3b_4, \quad \beta_9 = -\frac{8}{3\Omega^3}b_4b_3 - \frac{16}{\Omega^3}a_4b_4 - \frac{40}{\Omega^3}b_3b_5,$$
$$\beta_{10} = -\frac{4}{\Omega^3}b_3^2 - \frac{4}{3\Omega^3}a_3b_3 - \frac{8}{3\Omega^3}b_4b_5 - \frac{8}{\Omega}a_7b_4, \quad \beta_{11} = -\frac{28}{3\Omega^3}b_3b_5 - \frac{8}{3\Omega^3}a_4b_4. \tag{B.1}$$

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